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GENERALIZED NASH GAMES WITH SHARED CONSTRAINTS: EXISTENCE,
EFFICIENCY, REFINEMENT AND EQUILIBRIUM CONSTRAINTS

BY

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DISSERTATION

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Abstract

The thesis pertains to some fundamental questions in the theory of games. Our focus is on a class of noncooperative N -player games, called *generalized* Nash games with *shared constraints*, or simply, *shared-constraint games* [Ros65]. In such a game, every strategy-tuple is constrained to lie in a subset \mathbb{C} of the product space of strategies. Thus strategies available to a player are only those which when taken jointly with the strategies of all other players, form a tuple that lies in \mathbb{C} . The set \mathbb{C} is called the *shared constraint*.

Despite their relevance in real-world settings, there are many theoretical properties of these games that are not well understood. What interests us in this thesis is the theoretical character of the equilibria of these games. Shared-constraint games admit two *kinds* of equilibria: generalized Nash equilibria (GNE) that are ill-posed and often intractable and a smaller subset of them (called variational equilibria or VE) satisfying an exogenous regularity condition that are well-posed and surprisingly tractable. We seek to clarify the nature of these equilibria, study their economic implications and exploit their properties to advance the analytical theory for conventional and somewhat unconventional shared-constraint games. The unconventional shared-constraint games are in fact a class of *dynamic* games, called *multi-leader multi-follower games*, which we view through the lens of shared constraints.

Four questions are addressed in this thesis with the central theme as shared-constraint games. In the **first** part of the thesis we present a *refinement* [MWG95] of the GNE. A refinement of an equilibrium is a subset of the set of equilibria which is nonempty whenever the original set of equilibria is nonempty. Refined equilibria are representative of the set of equilibria and have additional properties that make them more attractive as solution concepts than the original equilibria. The contribution of this work is a theory that gives sufficient conditions for a game to admit the VE as a refinement of the GNE. These conditions are expressed in terms of the *Brouwer degree*, which is seen to relate the GNE and the VE in a profound manner. Importantly, for certain classes of games, these conditions are also seen to be necessary. The degree theoretic relationship holds in both, primal and primal-dual space. Our work unifies some previously known results and provides mathematical justification for ideas that were known to be intuitive appealing but were hitherto unsubstantiated formally.

The **second** part of this thesis is about multi-leader multi-follower games. These games are

highly nonconvex and irregular and no reliable theory is available for claiming the existence of equilibria of these games. We develop such a theory for multi-leader multi-follower games with shared constraints. The application of standard fixed point arguments to the reaction map of general multi-leader multi-follower games is hindered by the lack of suitable continuity properties, amongst other requirements, in this map. We observe that these games bear a close resemblance to shared-constraint games and present modifications of the canonical multi-leader multi-follower game that result in shared-constraint games, with far more favorable properties. Specifically, a global equilibrium of this game exists when a suitably defined *modified* reaction map admits a fixed point. Sufficient conditions for the existence of these fixed points are obtained via topological fixed point theory. Finally, the paradigm developed is applied to a class of LCP-constrained leader problems where conditions for the contractibility of the domain are derived via the theory of retracts.

The **third** part of thesis concerns the use of variational inequalities for claiming the existence of an equilibrium to shared-constraint games. The equilibrium conditions of a generalized Nash game can be compactly stated as a quasi-variational inequality (QVI), an extension of the variational inequality (VI). Harker [Har91] showed that under certain conditions on the maps defining the QVI, a solution to a related VI solves the QVI. But the application of Harker's result to the QVI associated with shared-constraint games proves difficult because its hypotheses can fail to hold even for simple shared-constraint games. We show these hypotheses are in fact impossible to satisfy in most settings. But we show that for a modified QVI, whose solution set equals that of the original QVI, the hypothesis of Harker's result always hold. This paves the way for applying this result to shared-constraint games, albeit in an indirect manner. This avenue allows us to recover as a special case, a result proved by Facchinei et al. [FFP07], in which it is shown that a suitably defined variational inequality provides a solution to the QVI of a shared-constraint game.

In the **fourth** part we take a system-level view of shared-constraint games that result from resource allocation. We clarify the relation between this mode of allocating resources and the other conventional modes via either perfect competition or through the use of a mechanism. We find that for perfectly competitive settings the VE is the same as the competitive equilibrium. We then compare the performance of GNE and VE of the shared-constraint game with respect to the system-level objective of maximization of social welfare [MWG95] or aggregate utility. We are specifically interested in the *efficiency* of an equilibrium, which is the ratio of the aggregate utility for this equilibrium to the optimal aggregate utility, and in the lowest value this efficiency can take for a class of utility functions. We show that for a certain class of utility functions the VEs are fully efficient. We characterize this class and show that departures from this setting can lead to arbitrarily low efficiency in the worst case. Specifically, in this class, even while the VEs are efficient, GNEs can be arbitrarily inefficient, and VEs of games not belonging to the 'efficient' class can have arbitrarily low efficiency in the worst case.

Abstract

Finally we suggest ways to remedy the low efficiency of equilibria in these cases. We find that a more restricted class of utility functions, in which the gradient map of every member utility function is bounded away from zero and from above uniformly over the domain, gives a more favorable worst case efficiency. We then consider a game where players incur costs that, from the system point of view are not additive, whereby the system problem is not merely the sum of the objectives of all players. We characterize utility functions for which the VE is efficient under this notion of efficiency. Finally we consider the imposition of a reserve price on players. The reserve price has the effect of eliminating players with low interest in the resource. The GNE is more indicative of the system optimal. We find that under certain conditions, efficiency as high as unity is obtainable by the imposition of an appropriate reserve price.

To Anupama

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Table of Contents

List of Symbols	x
Preliminaries	xii
Chapter 1 Introduction	1
1.1 Shared-constraint games	1
1.1.1 The canonical game and its equilibria	5
1.1.2 History	10
1.2 Refinement of the generalized Nash equilibrium	13
1.3 Global equilibria of multi-leader multi-follower games	14
1.4 Generalized Nash games and variational inequalities	16
1.5 Efficiency of GNEs of resource allocation games	17
Chapter 2 Refinement of the Generalized Nash Equilibrium	20
2.1 Introduction	20
2.1.1 Background	25
2.2 Primal generalized Nash and variational equilibria	27
2.2.1 The properties of K	27
2.2.2 Refinement of the generalized Nash equilibrium	28
2.2.3 Identification of subclasses of $\mathcal{S} \setminus \mathcal{S}'_2$	38
2.3 Primal-dual generalized Nash and variational equilibria	42
2.3.1 Refinement of the primal-dual GNE	45
2.4 Refinement of the GNE in power markets	51
2.5 Conclusions	54
2.6 Supplementary examples and results	54
2.6.1 Some examples of shared-constraint games	54
2.6.2 On the existence of a manifold of GNEs	56
2.6.3 Proof Lemma 2.2	57
2.6.4 Proof of Lemma 2.3	57
Chapter 3 Global Equilibria of Multi-leader Multi-follower Games	59
3.1 Introduction	59
3.1.1 Contributions	62
3.2 The EPEC as a shared-constraint game	65
3.2.1 Coupled constraint games	66
3.2.2 Shared constraint formulations	68

Table of Contents

3.2.3	Fixed point formulation through the modified reaction map	72
3.2.4	Properties of Υ	80
3.2.5	Comparison with the original EPEC	81
3.3	Nonconvex fixed point theorems	83
3.3.1	Fixed point property and absolute retracts	84
3.3.2	Fixed point theory for nonconvex set-valued maps	89
3.4	Fixed point theory for Υ	90
3.4.1	Broad results	91
3.4.2	Refined results and pathology	96
3.5	Contractibility of \mathcal{F}	98
3.5.1	EPECs arising from competing bilevel problems	98
3.5.2	EPECs with repeated equilibrium constraints	100
3.5.3	EPEC with consistent conjectures	102
3.6	Conclusions	103
Chapter 4 Generalized Nash Games and Variational Inequalities		104
4.1	Introduction	104
4.2	Harker's conditions	106
4.3	Applying Harker's conditions to the modified QVI	113
4.4	Conclusions	115
Chapter 5 The Efficiency of Generalized Nash Equilibria		117
5.1	Introduction	117
5.2	Preliminaries	120
5.3	Relation to past work	125
5.3.1	Relation to the competitive equilibrium	127
5.3.2	Relation to mechanism design	129
5.4	A general efficiency bound	130
5.4.1	Worst case efficiency of the VE	132
5.5	Games where the VE is efficient	135
5.5.1	Worst case efficiency of the GNE	139
5.6	Remedying zero worst case efficiency	142
5.6.1	Utility functions with bounded gradients	142
5.6.2	Other notions of efficiency	143
5.6.3	Reserve price	147
5.7	Conclusions	150
Appendix A Mathematical background		151
A.1	Convex analysis	151
A.2	Set-valued analysis	152
A.3	Variational inequalities	154
References		157

List of Symbols

Problem classes

EPEC	Equilibrium program with equilibrium constraints
LCP	Linear complementarity problem
MPEC	Mathematical program with equilibrium constraints
QVI	Quasi-variational inequality
VI	Variational inequality
SOL	$\text{SOL}(P)$ denotes the solution set of a problem P

Game theory

GNE	Generalized Nash equilibrium
VE	Variational equilibrium

Mathematics

$\mathbf{1}$	A vector with coordinate 1
2^Y	Set of all subsets of Y
AR	Absolute retract
ANR	Absolute neighbourhood retract
$\text{conv } A$	Convex hull of a set A
$\deg(f, \Omega, p)$	Degree of function f over set Ω with respect to p
$\mathbf{F}_V^{\text{nat}}$	Natural map of the variational inequality with set V
$\widetilde{\mathbf{F}}_L^{\text{nat}}$	Natural map of the variational inequality with set-valued map L

List of Symbols

$\mathcal{N}(z; X)$	Normal cone of a set X at a point $z \in X$
\mathbf{I}	Identity map on a domain
$\mathcal{T}(z; X)$	Tangent cone of a set X at a point $z \in X$
X^*	Dual cone of a cone X
X_∞	Recession cone of a set X
A^T	denotes the transpose of a matrix (or vector) A .
$\ x\ $	Euclidean norm of vector x
$\text{int}(A)$	Interior of a set A
∂A	Boundary of a set A
\overline{A}	Closure of a set A
$f \circ g$	Composition of function f with g
∇f	Gradient of function f
$\nabla_i f$	If f is a function of $x = (x_1, \dots, x_N)$, partial gradient with respect to x_i
$u \perp v$	For $u, v \in \mathbb{R}^n$, $u_i v_i = 0, i = 1, \dots, n$
$\Pi_V(z)$	Projection on the set V of a point z

Preliminaries

■ Notation

Most of the notation we use is explained at relevant places in the thesis. Some of the common notation is gathered in this section for easy reference. Abbreviations and notation for mathematical operations are explained in the List of Symbols.

The symbol \mathbb{R} stands for the real line, \mathbb{N} stands for the set of natural numbers and \mathbb{R}_+ stands for the interval $[0, \infty)$. \mathbb{R}^n and \mathbb{R}_+^n for a natural number n , stand for the n -fold product of \mathbb{R} and \mathbb{R}_+ , respectively. No complex numbers appear in this thesis, so a scalar mentioned without a reference to an ambient space is understood to be a real number. Thereby, we use the notation \mathbb{C} , commonly used for complex numbers, to denote a specific kind of set called a shared constraint that appears repeatedly in this thesis. Italicized letters, e.g., x are used to denote vectors (and scalars) and are, by default, column vectors. Letters i, j, k, ℓ, m, n are reserved for denoting integers and are often used as indices. In the case of summations or products, if the variable over which the operation is carried out is not explicitly mentioned, this variable is to be understood as the index.

Tuples occur ubiquitously in this thesis. Components of the tuple are subscripted by their index, while the tuples themselves are denoted without any subscripts. E.g., by x we denote the tuple (x_1, x_2, \dots, x_N) , where x_i denotes the i^{th} component of x . The notations x^{-i} and (y_i, x^{-i}) stand for the tuples

$$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \quad \text{and} \quad (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N),$$

respectively. Throughout, $\mathcal{N} = \{1, 2, \dots, N\}$ denotes a set of players, φ_i is used to denote the objective function of player i and Φ denotes the tuple $\{\varphi_1, \dots, \varphi_N\}$.

■ Prerequisites

Prerequisites for this thesis can be classified into those that are necessary and those that are recommended, but not absolutely necessary. This thesis is about the theory of games and, in particular, about some of its fundamental notions. Therefore a knowledge of the concept of a game and the Nash equilibrium [Nas50, Nas51] is necessary to read it. Familiarity with some of the more advanced

concepts, particularly the refinement of an equilibrium and the concept of leadership in games is would be beneficial. We briefly review these concepts, but our reviews cannot substitute a more thorough initiation through a book such that of Başar and Olsder [BO99]. In addition, it would be useful to know the concept economic efficiency, for which the book by Mas-Colell, Whinston and Green [MWG95] is a source.

The questions this thesis addresses are of game-theoretic origin but of mathematical character. The effort to answer them in with greater generality has necessitated, on occasions, the use of advanced tools, such as topological degree theory, the theory of retracts and topological fixed point theory. This thesis provides only a working knowledge of them. For a detailed discourse on degree theory, we recommend the books by Fonseca and Gangbo [FG95] and Kesavan [Kes04] and the books by Dugundji and Granas [DG03] and Borsuk [Bor67] for topological fixed point theory and the theory of retracts respectively. Given the extent of the mathematical content in thesis, some of the more basic results are taken for granted. A knowledge of real analysis and topology of metric spaces at the level of Rudin [Rud76] and of topological vector spaces at the level of Rudin [Rud91] ought to familiarize the reader with these basic results.

An appendix is included at the end which recapitulates a few results from convex analysis, set-valued analysis and variational inequalities.

■ Notes

By and large we use American spellings in our use of English. Versions of Chapters 2, 3 and 4 are currently under review with journals as articles [KS10b], [KS10a] and [KS10c], respectively.

Chapter 1

Introduction

■ 1.1 Shared-constraint games

The thesis pertains to some fundamental questions in the theory of games. Our focus is on a class of noncooperative N -player games, called *generalized* Nash games with *shared constraints*, or simply, *shared-constraint games*. In a *generalized* Nash game, the strategies available to a player are allowed to *depend* on the strategies of its adversaries. In a game with shared constraints, this dependence is of a specific kind: there is a set \mathbb{C} in the product space of strategies in which every strategy-tuple is constrained to lie. Thus strategies available to a player in this game are only those which when taken jointly with the strategies of all other players, form a tuple that lies in \mathbb{C} . The set \mathbb{C} is called the *shared constraint*.

The abstract game described above is in fact of substantial practical relevance since shared-constraint games are natural models for many real-world settings. For example, a game where players compete for portions of a resource such as bandwidth can be modelled as a shared-constraint game. In this context the shared constraint \mathbb{C} is the set of N -tuples of accessed bandwidths that together do not exceed capacity. Another example is a game in power markets where players decide the power to be sold at various nodes of a network while being subject to a common set of transmission constraints. Here \mathbb{C} is the set of flows permitted by the network topology and Kirchoff's laws. Generally speaking, a setting in which there is competition between players under an over-arching system-level requirement binding the choices of all players, is or can be modelled as a shared-constraint game.

Despite their practical relevance, there are many theoretical properties of these games that are not

well understood, which gives us an opportunity for fundamental research and the possibility of new findings. What interests us in this thesis is the theoretical character of the equilibria of these games. We seek to clarify the nature of these equilibria, study their economic implications and exploit their properties to advance the analytical theory for conventional and somewhat unconventional shared-constraint games. The unconventional shared-constraint games are in fact a class of *dynamic* games, called *multi-leader multi-follower* games which we view through the lens of shared constraints.

Classical Nash games correspond to the special case of a shared-constraint game where \mathbb{C} is a cartesian product of sets drawn from individual strategy spaces. On the other hand, in generalized Nash games *without* shared constraints the set \mathbb{C} differs from player to player. i.e., for every player i , there is a set \mathbb{C}_i , different from \mathbb{C}_j for some j , such that every N -tuple of strategies must lie in \mathbb{C}_i . Both these features – the non-cartesian nature of \mathbb{C} and that $\mathbb{C} = \mathbb{C}_i$ for all players i – endow the equilibria of shared-constraint games with some unique properties. Of particular importance to us in this thesis are the conditions that suffice for the existence of equilibria and the property of nonuniqueness of equilibria. Shared-constraint games yield equilibria under nearly the same conditions as classical Nash games. This is significantly different from the situation with generalized Nash games, which usually require markedly stronger assumptions. And while classical Nash games often yield unique equilibria, shared-constraint games have unique equilibria only in exceptional and somewhat pathological cases.

These properties stem from the underdeterminacy of the equilibrium conditions of a shared-constraint game. Shared-constraint games admit two *kinds* of equilibria: general equilibria that are ill-posed and often intractable and a smaller subset of them satisfying an exogenous regularity condition that are well-posed and surprisingly tractable. We discover in this thesis that these equilibria also have different economic consequences. For example, ill-posed equilibria have higher *informational requirements* compared to the well-posed equilibria for implementation. Ill-posed equilibria also suffer from higher *efficiency loss* compared to the well-posed equilibria. On the other hand, well-posed equilibria come with the burden of justifying the exogenous condition that generates them in economic terms.

The nature of these equilibria and their game-theoretic consequences provide a fertile ground for a fascinating theoretical inquiry in which the present thesis takes its roots. Based on our assumptions, questions in this thesis fall into two distinct categories. One category deals directly

1.1 Shared-constraint games

with shared-constraint games and has a more conventional setting, namely, where \mathbb{C} is a convex set. Here our contributions lie in defining a *refinement* for the equilibria of these games, facilitating their analysis through the use of *variational inequalities* and investigating the loss of efficiency, or departure from optimal centralized outcomes, that these equilibria display. The second category pertains to multi-leader multi-follower games. The equilibrium of such a game is given by a problem characterized by nonconvexity and lack of regularity for which no formal and general existence theory is known. These games are technically *not* shared-constraint games, but are observed to be closely related to them. By exploiting the analytical potential of the shared-constraint game, we develop a theory of existence of equilibria for multi-leader multi-follower games.

Four questions are addressed in this thesis whose treatment naturally separates out into the four chapters that follow the present chapter. In Chapter 2 of this thesis we ask the following question:

How representative are the ‘tractable’ equilibria and how does one justify the exogenous requirement that generates them in rigorous game-theoretic terms?

We present a *refinement* of the equilibrium concept for convex shared-constraint games. A refinement of an equilibrium is a subset of the set of equilibria which is nonempty whenever the original set of equilibria is nonempty. The refined equilibria are representative of the set of equilibria and have additional properties that make them more attractive solution concepts than the original equilibria. We give sufficient conditions for the well-posed equilibria to be a refinement of the more general ill-posed equilibria, whereby the former are representative of the latter.

The next part of this thesis, Chapter 3, is about multi-leader multi-follower games, with the following central question:

The class of multi-leader multi-follower games, for which no reliable existence theory is available, bear a close resemblance to games with shared constraints. Can one exploit this connection to provide an existence theory?

The constraint that the followers be in equilibrium appears in each leader’s optimization problem. We observe that, as a result of this constraint, these games bear a close resemblance to shared-constraint games. These games are in general highly nonconvex and irregular and are consequently very difficult to analyze. We provide a theory of existence of equilibria to such games that is based on the concept of a shared-constraint game.

Chapter 1. Introduction

Chapter 4 is about the use of variational inequalities for claiming the existence of an equilibrium to shared-constraint games.

An result by Harker [Har91] guarantees that, under certain hypotheses, the solution of a variational inequality is an equilibrium of the shared-constraint game. But this result has been observed to be hard to apply to such shared-constraint games because its hypotheses are difficult to satisfy. Can we clarify the reach of this result?

We investigate the applicability of this result to shared-constraint games and show that under the usual manner of application, its hypotheses are in fact impossible to satisfy in most settings. But we present an indirect way of applying the result, under which the hypotheses always hold, paving the way for its application to all shared-constraint games.

In Chapter 5 we take a system-level view of shared-constraint games that result from resource allocation.

Shared-constraint games can be used to model resource allocation settings wherein a mechanism or an administrator cannot be used. What is the economic efficiency of the equilibria that result from shared-constraint games of this kind?

We clarify the relation between resource allocation via shared-constraint games, via perfect competition and via the use of a mechanism. We compare the performance on the equilibria of the shared-constraint game in terms of their best case and worst case efficiency. Furthermore, characterize settings where the tractable equilibria are fully efficient and show that departures from this setting can lead to arbitrarily low efficiency in the worst case. Finally, we suggest ways to remedy the low efficiency seen in the worst case.

Mathematically speaking, our approach to these questions is of topological flavor and rooted in the view that fixed point theory is a rich way to probe the properties of the equilibria of games. But our arguments are developed and communicated using tools and constructs of mathematical programming, such as variational inequalities and its allied problems. On occasions we will make quick translations from equilibria of games to fixed point problems and mathematical programming problems. We hope that the introduction in the following section and the background in Appendix A will make the reader comfortable with these translations.

1.1 Shared-constraint games

In Sections 1.2, 1.3, 1.4 and 1.5, we present a synopsis of Chapters 2, 3, 4 and 5 respectively. To state the contributions with precision and to provide a historical perspective, we describe a canonical shared-constraint game and present a brief history of these games in Section 1.1.1 and Section 1.1.2.

■ 1.1.1 The canonical game and its equilibria

We now formally define a shared-constraint game. Let $\mathcal{N} = \{1, 2, \dots, N\}$ be a set of *players*. Players are characterized by a *strategy space* and an *objective function*. Let m_1, \dots, m_N be positive integers and $m = \sum m_i$. Let \mathbb{R}^{m_i} be player i 's strategy space and $x_i \in \mathbb{R}^{m_i}$ represent its strategy. If x denotes the tuple of strategies (x_1, x_2, \dots, x_N) , then in this game, each tuple x is required to lie in a set $\mathbb{C} \subseteq \mathbb{R}^m$. This set is the aforementioned *shared constraint*. The objective function of player i is a function $\varphi_i : \mathbb{C} \rightarrow \mathbb{R}$, which we assume they seek to minimize.

Suppose x^{-i} denotes the tuple

$$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$$

and (y_i, x^{-i}) the tuple

$$(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N).$$

Let $K_i : \mathbb{R}^{m-m_i} \rightarrow 2^{\mathbb{R}^{m_i}}$, for each $i \in \mathcal{N}$ and $K : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$ denote the following set-valued maps

$$K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N} \quad \text{and} \quad K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}). \quad (1.1)$$

In the canonical generalized Nash game with shared-constraint \mathbb{C} , player i is assumed to solve the parameterized optimization problem,

$\begin{aligned} & \text{A}_i(x^{-i}) && \underset{x_i}{\text{minimize}} && \varphi_i(x_i; x^{-i}) \\ & && \text{subject to} && x_i \in K_i(x^{-i}). \end{aligned}$

In this problem player i assumes x^{-i} as fixed and minimizes $\varphi_i(\cdot, x^{-i})$ over the set of those x_i that together with x^{-i} lie in \mathbb{C} , i.e. $\{x_i \mid (x_i, x^{-i}) \in \mathbb{C}\}$. Let $\Phi = (\varphi_1, \dots, \varphi_N)$ and $\mathbf{m} = (m_1, \dots, m_N)$.

Chapter 1. Introduction

By a *shared-constraint game* formed from the optimization problems $(A_1), \dots, (A_N)$, we mean the tuple $\mathcal{G} := (\mathcal{N}, \mathbf{m}, \Phi, \mathbb{C})$.

Note that the case of a player i *maximizing* a certain objective $\bar{\varphi}_i$ is covered in the above formulation by taking $\varphi_i \equiv -\bar{\varphi}_i$. Furthermore, constraints on players that are independent of the strategies of other players are also covered in this formulation, as follows. If each player i faces a constraint $x_i \subseteq X_i \subseteq \mathbb{R}^{m_i}$, where X_i does not depend on x^{-i} , in addition to a shared constraint $\mathbb{C}' \subseteq \mathbb{R}^m$, then the resulting game is equivalent to the game \mathcal{G} with $\mathbb{C} := \prod X_i \cap \mathbb{C}'$. This also shows that when \mathbb{C} is a cartesian product of sets in \mathbb{R}^{m_i} , the shared-constraint game \mathcal{G} is equivalent to a classical Nash game.

One may contrast the canonical shared-constraint game with a canonical generalized Nash game (*without* shared constraints). In this game player i is faced with optimization problem

$$\boxed{\begin{array}{ll} A'_i(x^{-i}) & \text{minimize}_{x_i} \quad \varphi_i(x_i; x^{-i}) \\ & \text{subject to} \quad x_i \in K'_i(x^{-i}), \end{array}}$$

where

$$K'_i(x^{-i}) = \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}_i\},$$

for some set $\mathbb{C}_i \subseteq \mathbb{R}^m$ and objective functions $\varphi_i : \mathbb{C}_i \rightarrow \mathbb{R}$ for each $i \in \mathcal{N}$.

The distinction between these classes can be seen in Fig 1.1. Each figure therein shows a game with two players with objectives φ_1 and φ_2 whose graphs are depicted “above” the domain, which is a subset of the product of strategy spaces of player 1 and player 2. The curved lines on these graphs are sections of φ_1 and φ_2 obtained by fixing strategies of player 2 and player 1 at x_2 and x_1 , respectively. The point (x_1, x_2) is shown by a bold dot in the domain. The set over which each player optimizes (the section of) its objective, i.e. the feasible region of its optimization problem, is shown by bold lines passing through this dot and parallel to the strategy space of the player. Technically, these lines are subsets of the strategy spaces of players and have been represented by translating them so as to pass through the point (x_1, x_2) . These classes of games differ by the requirements through which these lines are generated. In Fig 1.1(a), a classical Nash game is depicted wherein the set \mathbb{C} is a cartesian product of two sets X_1 and X_2 drawn from the

1.1 Shared-constraint games

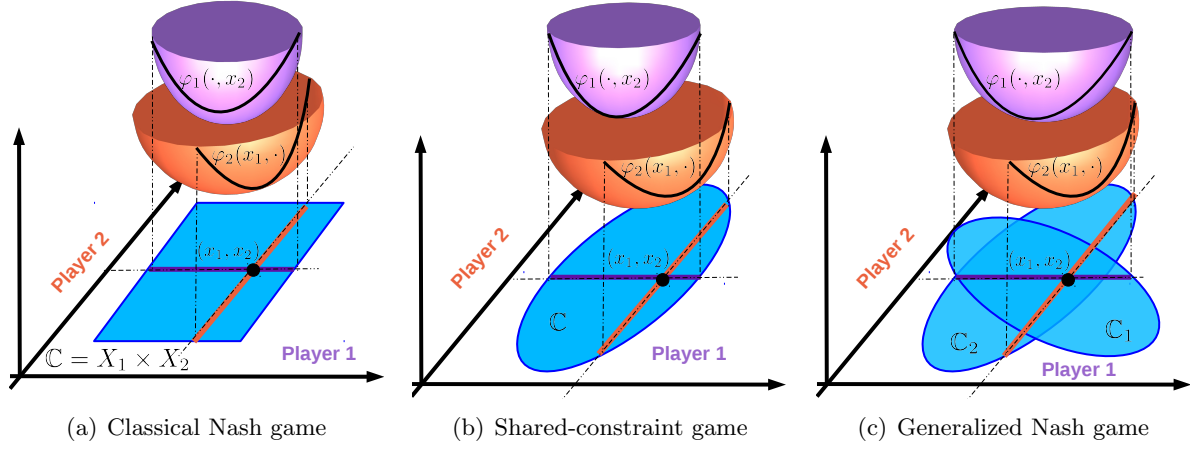


Figure 1.1: Classes of games

strategy spaces of player 1 and 2 respectively. In this case the feasible region of any player i 's optimization problem is independent of x^{-i} . In Fig 1.1(b), we see a shared-constraint game with shared constraint \mathbb{C} which is a subset of product of strategy spaces and is *not* cartesian set itself. Observe that the feasible region of player i now changes with x^{-i} and that these feasible regions are generated through the set \mathbb{C} . Fig 1.1(c) depicts a generalized Nash game with sets \mathbb{C}_1 and \mathbb{C}_2 . The feasible region of player 1 (resp. 2) is obtained as those strategies of player 1 (resp. 2) that together with x_2 (resp. x_1) lie \mathbb{C}_1 (resp. \mathbb{C}_2). When $\mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}$ we recover a shared-constraint game. When \mathbb{C} is a cartesian, we recover a classical Nash game.

The objective functions in our shared-constraint game are continuous; this reflects a continuous variation in the preferences of the players over their alternatives. It is somewhat appropriate in a game to take for each i the objective function φ_i to have as domain the graph of K_i , which is \mathbb{C} for the shared-constraint game (and the graph of K'_i , i.e. \mathbb{C}_i , in the generalized Nash game). Yet, it is common to assume that each objective function φ_i is in fact defined over all of \mathbb{R}^m . Broadly speaking, no generality is lost in this assumption when \mathbb{C} has a certain topological structure¹, for e.g. when \mathbb{C} is convex. In instances in this thesis where our thrust is not on technical generality but on insights, we take these domains to be \mathbb{R}^m .

Consider the game \mathcal{G} again. We now define a solution concept for this game as an extension of

¹The precise property needed is that \mathbb{C} be an *absolute retract*.

Chapter 1. Introduction

the Nash equilibrium.

Definition 1.1 (Generalized Nash equilibrium (GNE)) *A strategy tuple x is a generalized Nash equilibrium of \mathcal{G} if $x_i \in \text{SOL}(A_i(x^{-i}))$ for all $i \in \mathcal{N}$.*

Thus x is a GNE if and only if it is a fixed point of the following *reaction map* $\mathcal{R} : \mathbb{C} \rightarrow 2^{\text{range}(K)}$.

$$\mathcal{R}(x) = \{ \bar{x} \in K(x) \mid \varphi_i(\bar{x}_i, x^{-i}) \leq \varphi_i(u_i, x^{-i}) \quad \forall u_i \in K_i(x^{-i}) \text{ and } \forall i \in \mathcal{N} \}. \quad (1.2)$$

This fixed point problem expressed through \mathcal{R} characterizes the GNE, but is often analytically intractable. On the other hand consider the following fixed point problem. Let $\Psi : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$ be given by

$$\Psi(x, \bar{x}) = \sum_{i=1}^N \varphi_i(\bar{x}_i, x^{-i}) \quad \forall x, \bar{x} \in \mathbb{C}.$$

and consider the *modified reaction map* $\Upsilon : \mathbb{C} \rightarrow 2^{\mathbb{C}}$, defined as

$$\Upsilon(x) := \left\{ \bar{x} \in \mathbb{C} \mid \Psi(x, \bar{x}) = \inf_{u \in \mathbb{C}} \Psi(x, u) \right\}. \quad (1.3)$$

Rosen in [Ros65] showed that every fixed point of Υ is a fixed point of \mathcal{R} (though the converse is not true). The fixed point problem in Υ is significantly more tractable. The GNE is what we called in the introduction as the ‘ill-posed’ equilibrium, whereas fixed points of Υ are the ‘well-posed’ equilibria.

The disparity between these equilibria can be better understood in the case where \mathbb{C} is a closed convex set and for each i $\varphi_i(x)$ is a continuous differentiable function which is convex in x_i . In this case, x is a GNE of \mathcal{G} if and only if it solves the *quasi-variational inequality* (QVI) below.

Find $x \in K(x)$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in K(x).$ (QVI(K, F))
--

where $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the function given by

$$F(x) = \begin{pmatrix} \nabla_1 \varphi_1(x) \\ \vdots \\ \nabla_N \varphi_N(x) \end{pmatrix} \quad \forall x \in \mathbb{R}^m,$$

1.1 Shared-constraint games

and ∇_i is the partial derivative $\frac{\partial}{\partial x_i}$. It is easy to see that $x \in \text{SOL}(\text{QVI}(K, F))$ if and only if $x \in \mathcal{R}(x)$. The variational equilibrium (VE) is a particular kind of GNE defined in [FK07, FP09]:

Definition 1.2 (Variational equilibrium (VE)) *A strategy tuple x is said to be a variational equilibrium of \mathcal{G} if x is a solution of $\text{VI}(\mathbb{C}, F)$.*

Find $x \in \mathbb{C}$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in \mathbb{C}.$ (VI(\mathbb{C}, F))
--

The notation $\text{VI}(\mathbb{C}, F)$ stands for a *variational inequality*. A background on variational inequalities and quasi-variational inequalities is given in Appendix A.3. It is easy to show that the VE is in fact a fixed point of Υ . This is compatible with a result of Facchinei et al. [FFP07] where they showed that every VE is a GNE.

Theorem 1.1 (Every VE is a GNE) *Every solution of $\text{VI}(\mathbb{C}, F)$ is a GNE of \mathcal{G} .*

When \mathbb{C} can be described by an continuously differentiable algebraic function, the VE can also be interpreted in terms of the KKT systems, as observed by Facchinei et al. [FFP07].

Theorem 1.2 (Intepretation of the VE in terms of Lagrange multipliers)

Let $\mathbb{C} = \{x \in \mathbb{R}^m \mid x \geq 0, c(x) \geq 0\}$ for some concave continuously differentiable constraint function $c : \mathbb{R}^m \rightarrow \mathbb{R}^n$. x be a GNE for which the system of KKT conditions $\{\text{KKT}_i\}_{i \in \mathcal{N}}$

$$\begin{aligned} 0 &\leq x_i \perp \nabla_i \varphi_i(x) - \lambda_i \nabla_i c(x) \geq 0 \\ 0 &\leq \lambda_i \perp c(x) \geq 0, \end{aligned} \tag{KKT}_i$$

is satisfied with $\lambda_1 = \lambda_2 = \dots = \lambda_N$. Then x solves $\text{VI}(\mathbb{C}, F)$. Conversely if x solves $\text{VI}(\mathbb{C}, F)$ then there exist $\lambda \in \mathbb{R}^n$ such that $\{\text{KKT}_i\}_{i \in \mathcal{N}}$ hold with $\lambda_i = \lambda$ for all $i \in \mathcal{N}$.

The difference in the tractibility of the GNE and the VE can be seen more clearly in this form. Due to the repetition of the constraint $c(\cdot) \geq 0$ in each player's optimization problem, there are N Lagrange multipliers each of which is orthogonal to the shared constraint. The system of equations $\{\text{KKT}_i\}_{i \in \mathcal{N}}$ is thus underdetermined. The VE on the other hand satisfies the regularity inducing condition that all Lagrange multipliers corresponding to the shared constraint be *equal*.

If at a GNE x , we have $c(x) > 0$, then the all Lagrange multipliers are 0 and hence equal. Thus every GNE that lies in the interior of \mathbb{C} is also a VE, and therefore the distinction between the VE and the GNE is only on the boundary of \mathbb{C} . We see that the underdeterminacy of the KKT conditions is reflected in the dimensionality of certain *normal cones*. To demonstrate this assume that $\mathbb{C} = \{x \mid c(x) \geq 0\}$, for a concave, continuously differentiable $\mathbb{R}^m \rightarrow \mathbb{R}$ function c (i.e. \mathbb{C} doesn't include any nonnegativity constraints on the strategies). Recall (cf. Appendix A.3) that x is a VE if and only if $-F(x)$ lies in the normal cone $\mathcal{N}(x; \mathbb{C})$. Likewise x is a GNE if and only

$$-F(x) \in \mathcal{N}(x; K(x)) = \prod_{i \in \mathcal{N}} \mathcal{N}(x_i; K_i(x^{-i})).$$

Now, using a well known result from convex analysis [Roc97] (cf. Appendix A.1) we get

$$\mathcal{N}(x; \mathbb{C}) = \{-\alpha \nabla c(x) \mid \alpha \geq 0\} \quad \text{and} \quad \mathcal{N}(x_i; K_i(x^{-i})) = \{-\alpha \nabla_i c(x) \mid \alpha \geq 0\}, \quad \forall i \in \mathcal{N}.$$

We see that $\mathcal{N}(x; \mathbb{C})$ is a single dimensional ray, while $\mathcal{N}(x; K(x))$ is, in general, an N -dimensional cone. $\mathcal{N}(x; K(x))$ includes but is larger than $\mathcal{N}(x; \mathbb{C})$. This is illustrated in Fig 1.2 for a convex set \mathbb{C} in \mathbb{R}^2 with a continuously differentiable boundary, and $m_1 = m_2 = 1$ and $N = 2$. Fig 1.2 shows a point x on the boundary of \mathbb{C} . The shaded area to the north-west of x is $\mathcal{N}(x; K(x))$ while the ray normal to $\partial\mathbb{C}$ at x is $\mathcal{N}(x; \mathbb{C})$ (both with origin shifted to x .) The vector $-F(x)$ lies in $\mathcal{N}(x; K(x))$ but not in $\mathcal{N}(x; \mathbb{C})$.

This completes the background for shared-constraint games and their equilibria. The points made here form the basis for our research on the topic. More advanced conclusions based on these ideas are obtained in the chapters following the present chapter. We end this chapter with a short review of the history of shared-constraint games.

■ 1.1.2 History

An excellent historical perspective of the many developments in shared-constraint games can be found in the two extensive surveys by Facchinei and Kanzow [FK07, FK10]. We do not aim to be as extensive in our survey, but only to point out some important milestones in this history.

While games with strategic coupling in the constraints of players can be traced to the early

1.1 Shared-constraint games

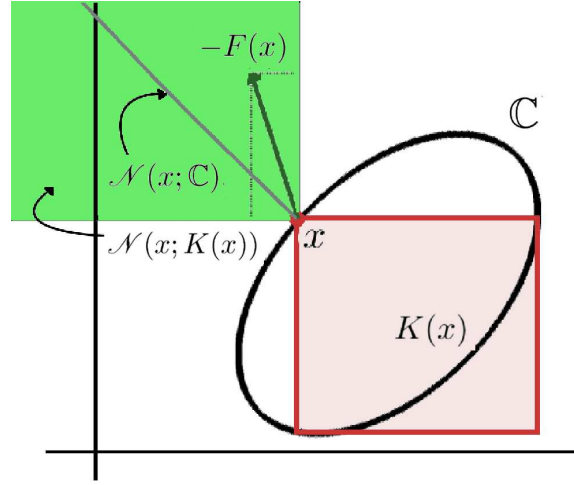


Figure 1.2: A point x on the boundary of \mathbb{C} that is a GNE but not a VE. Also shown are normal cones $\mathcal{N}(x; \mathbb{C})$ and $\mathcal{N}(x; K(x))$ (with their vertices shifted to x) and the vector $-F(x)$ (with origin shifted to x). $-F(x)$ belongs to $\mathcal{N}(x; K(x))$ but not to $\mathcal{N}(x; \mathbb{C})$.

years of game theory itself, games with shared constraints are a much later development. Classical matrix games were introduced by von Neumann and Morgenstern [vM44] in 1944 and studied by Nash [Nas50] in 1950. In 1952 Debreu [Deb52] studied games in which strategy sets were dependent on the strategies of other players and gave a theorem for the existence of an equilibrium to these games, calling it the *social equilibrium*. Arrow and Debreu in 1954 [AD54] developed the theory of these games in a more detailed manner. These games were then called *abstract economies*. The more specific case of a shared constraint was investigated much later in 1965 by Rosen [Ros65]. Rosen called them *coupled constraint games*. In the Operations Research community the name “shared-constraint games” is frequently used, though it has not entirely replaced the other names such as “Rosen-type games” or “coupled constraint games”. We prefer the name “shared-constraint games” since it more fittingly conveys the commonality of the constraint in question. The name “generalized Nash games” though has become a significantly more common term for “abstract economies”.

Today generalized Nash games are seen in a broad variety of settings outside their traditional setting in Economics and Operations Research. The earliest such examples are perhaps Robinson’s twin papers in 1993 [Rob93a, Rob93b] where the problem of measuring effectiveness in optimization-based combat models is studied through generalized Nash games. In the 1990’s and 2000’s, the

market-based allocation of resources in engineering systems began to grow in prominence. With the growing influence of game theory on these systems, generalized Nash games and shared-constraint games were also seen at the intersection of Economics and Engineering. Some work [AB02] on pricing and congestion control in communication networks and games in power markets [HMP00, CHH97, YAO08] are examples of the same.

We now give a short overview of the history of mathematical characterization of equilibria of games. That equilibria of games are solutions of fixed point problems is well known and quite easy to recognize. Indeed, Nash, in two separate papers proved the existence of the equilibrium through a fixed point argument based on Brouwer's [Nas51] and Kakutani's fixed point theorem [Nas50]. The fixed point formulation in terms of the reaction map \mathcal{R} in (1.2) perhaps appeared first in Arrow and Debreu's paper [AD54]. The formulation through Υ in (1.3) is due to Rosen [Ros65]. Many papers have since been written that analyze equilibria of games from this perspective and the effort has been to develop progressively weaker sufficient conditions for existence of equilibria. See, e.g., Shafer and Sonnenschein [SS75], Tsefatsion [Tes83], McClendon [McC86] and Tarafdar [Tar91].

Game theory has shared a symbiotic relationship with the theory of optimization. The most beautiful evidence of this is perhaps von Neumann's minimax theorem [vM44] for zero sum games which is the foundation for what we know today as linear programming duality. Indeed some the founders of optimization, like Kuhn [Kuh53, Kuh97] and Tucker² [Tuc60], were also game theorists. In the following years mathematical programming and game theory have seen many overlaps. The connection between (Q)VIs and (generalized) Nash equilibria appears to have been first observed by Bensoussan [Ben74] in 1974. One of the earliest attempts at using mathematical programming to *compute* the equilibria of games was made by Murhpy, Serali and Soyster [MSS82] 1982 wherein a simple Nash-Cournot model is studied. The use of VIs for this purpose was brought out with greater clarity by Harker in [Har84] in 1984. Since then there has been extensive work on computing the equilibria and studying the properties of oligopolistic markets through the tools and results of mathematical programming. Chapter 1.9 in [FP03] provides a thorough overview of the developments in this area.

The QVI was first introduced by Bensoussan, Goursat and Lions [BGL73] in 1973 and is a significantly harder problem than the VI. In 1991, Harker [Har91] demonstrated the use of QVIs in

²Tucker was also John Nash's advisor.

1.2 Refinement of the generalized Nash equilibrium

generalized Nash games and showed that under certain hypotheses a VI could be used to analyze generalized Nash games. Another important result of this flavor was proved in 2007 by Facchinei et al. [FFP07] in the context of shared-constraint games where they showed that every VE is a GNE. The results of Facchinei et al. and Harker mark the beginning of our work on this topic. These works relate solutions of problems to one another and build further theory based on such results. This thesis is driven by the search for similar results of game-theoretic nature and the construction of theories based on them. We rely on fixed point theory and the tools of mathematical programming in this search.

■ 1.2 Refinement of the generalized Nash equilibrium

The domain of this work is that of a closed and convex shared constraint \mathbb{C} . Our goal is to show that the VE is a *refinement* of the GNE for a wide class of shared-constraint games. A refinement of the set of equilibria of a game is a subset satisfying a specified rule, where this rule has the property that any game with a nonempty set of equilibria also possesses an equilibrium satisfying this rule (i.e. the refinement of the equilibrium is a subset of the equilibria which is nonempty whenever the equilibrium exists). Both the refined equilibria and the rule generating them are collectively referred to as the *refinement*.

This question is important from various standpoints. Firstly, games such as \mathcal{G} (from Section 1.1.1) are known to admit a large number of GNEs. Consequently, there is a need to define a refinement of the GNE that will retain only the meaningful GNEs. Secondly, the VE is easier to characterize and compute than a GNE. In general, obtaining a GNE requires a solution of the ill-posed system $\{\text{KKT}_i\}_{i \in \mathcal{N}}$ which leads to a *quasi-variational inequality* in the primal-space and a non-square complementarity problem in the primal-dual space. The VE, on the other hand, requires the solution of either a variational inequality (primal space) or a square complementarity problem (primal-dual space) both of which being far more tractable objects. So in attempting to show that a GNE exists, one may choose to focus on showing that a VE exists. Such a direction is sensible only when it is known that the existence of a GNE is sufficient for the existence of a VE. Thirdly, in many games with shared constraints, such as a bandwidth sharing game, the Lagrange multiplier corresponding to the constraint can be interpreted as the “price” charged on

a player for using a particular strategy by an administrator who controls the shared constraint. Thus the equilibrium with non-shared multipliers can be interpreted as an equilibrium resulting from *discriminatory prices*, i.e. prices that are charged by an administrator who may discriminate between various users of his constraint. The VE is an equilibrium with *uniform prices*. Often the situation modeled makes it unrealistic for the administrator to be able to distinguish between users and the economically appropriate equilibrium is one in which the *same* price is charged to all users, i.e. a VE is the only meaningful solution concept for these games.

The contribution of this work is a theory that gives sufficient conditions for a game to have the property that the existence of a GNE implies the existence of a VE. These conditions are expressed in terms of the *Brouwer degree*, which is seen to relate the GNE and the VE in a profound manner. Importantly, for certain classes of games, these conditions are also seen to be necessary. The degree theoretic relationship holds in both, primal and primal-dual space. Our work unifies some previously known results and provides mathematical justification for ideas that were known to intuitive appealing but were hitherto unsubstantiated formally.

■ 1.3 Global equilibria of multi-leader multi-follower games

This work pertains to the equilibria of a challenging class of games called multi-leader multi-follower games which we view through the lens of shared-constraint games. A multi-leader multi-follower game has players that are divided into categories of leaders and followers. Followers compete amongst each other in a conventional Nash game, assuming the decisions of leaders as fixed. Leaders compete against each other subject to an equilibrium amongst the followers. The equilibrium amongst the followers forms an *equilibrium constraint* in the optimization problems of the leaders. Thus the equilibrium amongst leaders is the solution of an **equilibrium program** with equilibrium constraints, or EPEC (cf. [PF05, Su05, DX09, HR07, Su07, Out04]).

The canonical EPEC has leaders whose optimization problems are of the following kind.

$$\begin{array}{ll}
 \mathbf{L}_i(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \\
 & \text{subject to} \quad (x_i, y_i) \in X_i \times Y_i \\
 & \quad y_i \in \text{SOL}(\text{VI}(F(x_i, x^{-i}, \cdot), K(x_i, x^{-i}))),
 \end{array}$$

1.3 Global equilibria of multi-leader multi-follower games

where $x_i \in \mathbb{R}^{m_i}$ is leader i 's strategy and y_i is the tuple of follower strategies that form the equilibrium of a Nash game parameterized by the tuple $x = (x_1, \dots, x_N)$, of the strategies of the N leaders. For each x , follower equilibria are characterized by the solution of the variational inequality $\text{VI}(F(x, \cdot), K(x))$. We denote this EPEC by \mathcal{E} , by $\Omega_i(x^{-i}, y^{-i})$, the feasible region of $L_i(x^{-i}, y^{-i})$ and by \mathcal{F} , the set of fixed points of

$$\Omega := \prod_{i=1}^N \Omega_i. \quad (1.4)$$

By a *leader-follower Nash equilibrium*, or simply equilibrium, of this game is meant a tuple of leader-follower strategies $(x, y) \in \mathcal{F}$ such that

$$\varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(u_i, v_i; x^{-i}) \quad \forall (u_i, v_i) \in \Omega_i(x^{-i}, y^{-i}), \quad i = 1, \dots, N.$$

Notice that this is in contrast with the usual approach to MPECs of using first order conditions.

Apart from its importance in modeling, the EPEC is also of theoretical interest. The central theoretical open question pertaining to EPECs is that of existence results, and indeed, to find a mathematical *principle* on which the existence of an equilibrium to EPEC may rest. In conventional Nash games with convex strategy sets such principles are well known – the fixed point theorems of Brouwer and Kakutani. But since convexity rarely holds for the equilibrium constraints and the presence of the opponent's strategies in these constraints, these theorems do not apply to EPECs directly. Consequently, while the need to analyze equilibria of EPECs is imminent, there currently exists no paradigm for this purpose that can be expressed and understood in terms of mathematical concepts.

In this work we develop a such a paradigm based on the concept of a *shared-constraint game*. The EPEC does bear a strong resemblance to shared-constraint games because the equilibrium amongst followers is a constraint that appears in each leader's optimization problem. However, it turns out the conventional model of the EPEC, in which leader solves L_i , is technically not a shared-constraint game. This work presents *modifications* of this model that result in an EPEC with shared constraints. These modifications are economically meaningful and could be used to model multi-leader multi-follower competition.

An EPEC with shared constraints offers many analytical benefits which are not available in a

general EPEC. A sufficient condition for the existence of an equilibrium to a shared-constraint EPEC is the existence of a fixed point to a set-valued map which has properties that make it amenable for the application of fixed point theorems. The only challenge is that this map has a possibly nonconvex domain and takes nonconvex values whereby common fixed point theorems like Brouwer's or Kakutani's don't apply. We address this challenge by using *topological fixed point theory* and the *theory of retracts*.

The theoretical message of our work is that (a) a shared-constraint formulation of multi-leader-multi-follower games has much more analytical tractability and (b) *contractibility*, as opposed to convexity, is a key property for the existence of equilibria to EPECs obtained in this way. Since topological fixed point theory is very general and it contains in it the convex fixed point theorems of Brouwer and Kakutani, what results is a very general, and to a great extent, unifying theory of the existence of equilibria, that contains within it the already known theory of equilibria for convex constrained games. We also give sufficient conditions for LCP constrained EPECs to satisfy the conditions warranted by topological fixed point theory.

■ 1.4 Generalized Nash games and variational inequalities

It is well known that QVIs are harder to analyse than VIs, particularly to claim the existence of a solution to. In [Har91], Harker examined generalized Nash games through QVIs and gave a result that reduced, under certain conditions, the analytical challenge QVI to that of a VI. Harker showed that if a set satisfying certain conditions exists, a solution to a VI defined over this set provides a solution to the QVI. Following is Harker's result.

Theorem 1.3 (Harker's result) *Let F and L be respectively point-to-point and point-to-set mappings from \mathbb{R}^m to itself. Suppose that there exists a nonempty closed, convex set A such that*

- (i) $L(x) \subseteq A$ for all $x \in A$ and
- (ii) $x \in L(x)$ for all $x \in A$.

Then any solution to the variational inequality $\text{VI}(A, F)$ is a solution of $\text{QVI}(L, F)$

But, when this result is applied to shared-constraint games, a curious difficulty surfaces: even for the simplest of shared constraints, a set satisfying Harker's conditions appears very hard to find.

1.5 Efficiency of GNEs of resource allocation games

Motivated by this observation, we revisit Harker's conditions in depth to qualify their applicability to shared-constraint games. We show that for most shared-constraint games, if Harker's result is applied directly to them, Harker's conditions turn out to be such that they hold only when the shared-constraint is trivial, i.e. a cartesian product of sets. As a result, we are able to formally explain why it remains a challenge to find a set satisfying Harker's conditions.

However we also show applicability of Harker's result to shared-constraint games can be salvaged by an indirect route. Specifically, we construct a modified QVI that admits the same solution set as the original QVI, and for this modified QVI, we there always exists a set satisfying Harker's conditions. In short, Harker's result applies to all nontrivial shared constraint games when it is applied through the modified QVI, though it fails when applied directly. As a corollary we also see that Harker's result *implies* a result provided by Facchinei et al. [FFP07] as an alternative to Harker's result.

Any results that reduce the analysis of a QVI to a VI are particularly important and those provided by Harker and Facchinei et al. are, to the best of our knowledge, the only ones that enable such a reduction. This work clarifies their reach, and reveals a surprising relationship between these two results.

■ 1.5 Efficiency of GNEs of resource allocation games

This work considers shared-constraint games which arise from the competition for a finite resource, such as bandwidth or energy. The shared constraint is the common requirement that the portions of the resource allocated to all players must total to not exceed the available quantity of the resource. In particular, we are interested in the efficiency of the equilibria of this game. We say that an allocation is *efficient* if it implements the *aggregate* utility of all players and we call the *efficiency* of an allocation, the ratio of the aggregate utility for this allocation to that of an efficient allocation.

Our game may be thought of as a *resource allocation game*, though this term has acquired a somewhat more specific meaning that does not commonly allude to our setting. The traditional resource allocation game has players that compete in a noncooperative fashion to access the resource, but their competition is induced by a *mechanism*. Our game represents a departure from this approach. Our game is relevant if one makes the assumption that the option of a mechanism

does not exist. In our setting, players move simultaneously, in a noncooperative manner and compete for portions of the resource. A shared-constraint game naturally fits such an interaction. The goal behind our work is not to suggest this game as an alternative to the mechanism design driven approaches to resource allocation, but instead to analyze the specific setting where it is relevant and the efficiency of equilibria that result from it. The subtlety though is that shared-constraint games admit two kinds of solution concepts, generalized Nash equilibrium (GNE) and the variational equilibrium (VE), with two different economic interpretations. In the light of this, our goal is also to do a comparative study of these equilibria with respect to the metric of efficiency. Though our work is not intended as an alternative to other mechanism-based approaches to resource allocation, it contributes to identifying settings where the VEs of this game have higher efficiency than the equilibria induced by mechanisms.

We discuss the efficiency of these equilibria separately. In particular we are concerned with the worst case efficiency of the GNE and the VE over certain classes of functions. Ours is, to the best of our knowledge, the only work on the efficiency of equilibria in general shared-constraint games, though the social welfare of the equilibria of these games has been considered in the setting of congestion control [AB02]. We consider two kinds of utility functions. In the first kind, the utility derived by any player is a function only of the allocation it receives. In the second, and more general setting, utilities are dependent on allocations received by other players. The first setting has an interesting interpretation in terms of the *competitive* equilibrium. We also see that the VE is efficient in this setting, whereas the GNE can be arbitrarily inefficient.

In the more general case we are concerned with the best case efficiency and worst case efficiency over a class of utility functions. We characterize utility functions under which unit efficiency is obtained for the VE. We also show that a departure from this setting can lead to arbitrarily low efficiency. Specifically, if one considers the GNE as a solution concept, for settings where the VE is efficient, one can get arbitrarily low efficiency. And a departure from the “efficient” setting of the VE can lead to arbitrarily low efficiency for the VE.

We then suggest some ways in which this low efficiency may be remedied. We find that a more restricted class of utility functions in which the gradient map of every member utility function is bounded away from zero and from above uniformly over the domain, gives a more favorable worst case efficiency. We then consider an alternative notion of efficiency. Specifically, we consider a game

1.5 Efficiency of GNEs of resource allocation games

where players incur costs that, from the system point of view are not additive, whereby the system problem is not merely the sum of the objectives of all players. We characterize utility functions for which the VE is efficient under this notion of efficiency. Finally we consider the imposition of a reserve price on players. As a result of the reserve price, players with low interest in the resource are, in a sense, eliminated from the game and the GNE is more indicative of the system optimal. We find that under certain conditions, efficiency as high as unity is obtainable by the imposition of an appropriate reserve price.

Chapter 2

Refinement of the Generalized Nash Equilibrium

■ 2.1 Introduction

This chapter concerns noncooperative N -player *generalized Nash games* [Har91] where players are assumed to have continuous strategy sets that are *dependent* on the strategies of their adversaries. Such games represent generalizations of classical noncooperative games that have traditionally allowed for strategic interactions between players to be expressed only through their objective functions. In a frequently encountered class of generalized Nash games, player strategies are required to satisfy a *common* coupling constraint. These games are called generalized Nash games with *shared constraints* [Ros65] or simply shared-constraint games.

Let $\mathcal{N} = \{1, 2, \dots, N\}$ be a set of players, m_1, \dots, m_N be positive integers and $m = \sum_{i=1}^N m_i$. For each $i \in \mathcal{N}$, let $U_i \subseteq \mathbb{R}^{m_i}$ represent player i 's strategy set, $x_i \in U_i$ be his strategy and $\varphi_i : \mathbb{R}^m \rightarrow \mathbb{R}$ be his objective function. We use the following notation: by x we denote the tuple (x_1, x_2, \dots, x_N) , x^{-i} denotes the tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ and (y_i, x^{-i}) the tuple $(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N)$. A *shared constraint* is a requirement that the tuple x be constrained to lie in a set $\mathbb{C} \subseteq \mathbb{R}^m$. In the generalized Nash game with shared constraint \mathbb{C} , player i is assumed

2.1 Introduction

to solve the parameterized optimization problem,

$$\boxed{\begin{array}{ll} A_i(x^{-i}) & \text{minimize}_{x_i} \quad \varphi_i(x_i; x^{-i}) \\ & \text{subject to} \quad x_i \in K_i(x^{-i}), \end{array}}$$

where for each $i \in \mathcal{N}$ the set-valued maps $K_i : \prod_{j \neq i} \mathbb{R}^{m_j} \rightarrow 2^{\mathbb{R}^{m_i}}$ and the map $K : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$, are defined as

$$K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N} \quad \text{and} \quad K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}) \quad \forall x \in \mathbb{R}^m. \quad (2.1)$$

For simplicity, we have dropped the sets U_i in the above optimization problems and have assumed that \mathbb{C} is contained in $\prod_{i \in \mathcal{N}} U_i$. We denote the resulting game resulting from the above optimization problems by \mathcal{G} . The solution concept applied to analyze such games is called the *generalized Nash equilibrium* (GNE).

Definition 2.1 (Generalized Nash equilibrium (GNE)) *A strategy tuple x is a generalized Nash equilibrium of \mathcal{G} if $x_i \in \text{SOL}(A_i(x^{-i}))$ for all $i \in \mathcal{N}$.*

Here $\text{SOL}(P)$ refers to the solution set of an optimization problem P . The GNE is an extension of the *social equilibrium* proposed by Debreu [Deb52]; see also [Ros65, AD54] and the recent survey [FK07] for more on this. We now introduce another solution concept. The variational equilibrium (VE) is a specific kind of GNE defined in [FK07, FP09]:

Definition 2.2 (Variational equilibrium (VE)) *A strategy tuple x is said to be a variational equilibrium of \mathcal{G} if x is a solution of $\text{VI}(\mathbb{C}, F)$.*

The notation $\text{VI}(\mathbb{C}, F)$ denotes a *variational inequality* with mapping F and a set \mathbb{C} (see Section 2.1.1), where $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the function given by

$$F(x) = \left(\nabla_{x_1} \varphi_1(x)^T \quad \dots \quad \nabla_{x_N} \varphi_N(x)^T \right)^T \quad \forall x \in \mathbb{R}^m.$$

The goal of this work is to provide a theory for the VE to be a *refinement* of the GNE. From an economic standpoint, the notion of *refinement of an equilibrium* is rooted in the belief that the

Chapter 2. Refinement of the Generalized Nash Equilibrium

concept of an equilibrium may be far too weak to serve as a solution concept. Therefore, for a stronger solution concept, one avenue lies in refining the equilibrium concept in games. Given such a motivation, what properties are of relevance in constructing such a refinement? If the weakness of the original concept is on the count that certain equilibria have less economic justification, then a refinement should formalize this by excluding such equilibria. Naturally, for a refinement to be useful, it should lead to a nonempty set of equilibria when the original solution concept admits equilibria. Thus, a refinement of the set of equilibria of a game is (a) a *subset* satisfying a certain *rule*, where this rule has the property that (b) any game with a nonempty set of equilibria also possesses an equilibrium satisfying this rule. Both the refined equilibria and the rule generating them are collectively referred to as the refinement. Refinements of equilibria are considered in detail by Myerson [Mye97] and have been previously sought for a host of solution concepts in both static and dynamic games [BO99]. For instance, the subgame-perfect Nash equilibrium is a refinement of the Nash equilibrium of a dynamic game (see [NRTV07, ch. 3.8]); trembling hand perfect [Sel75] and proper [Mye78] equilibria are refinements of mixed Nash equilibria in static finite strategy games [BO99, Wei97].

It is known from [FFP07] that every VE is a GNE. Thus this work focuses on showing that, under suitable conditions, the existence of a GNE implies the existence of a VE, i.e. (b). There are at least two motivations for studying this question which we describe below. First, for a modeler (studying, say, traffic flow or bandwidth allocation), equilibria of a game can be regarded as “outcomes” that would result if the game were to be played out in practice and when a game has many equilibria there is no clear indication of this outcome. GNEs of games such as \mathcal{G} have properties that, we believe, warrant a refinement. These games are known to admit a large number, and in some cases, a manifold of GNEs (see [FK07]; also Theorem 2.17 in Section 2.6.2). In fact, in the following example, *every* strategy tuple in \mathbb{C} is a GNE.

Example 2.1. Game where every strategy tuple is a GNE: Consider a game where player i has real valued strategies and solves

$A_i(x^{-i})$	$\begin{aligned} & \underset{x_i}{\text{minimize}} && x_i \ell(X) \\ & \text{subject to} && X = \alpha \quad : \lambda_i, \end{aligned}$
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2.1 Introduction

where $X = \sum_{i \in \mathcal{N}} x_i$. Games such as this arise commonly in network routing. The Karush-Kuhn-Tucker (KKT) conditions characterizing the GNE, x^* , of this game are

$$(x_i^* \ell(X^*))' = \lambda_i, \quad \forall i \in \mathcal{N} \quad \text{and} \quad X^* = \alpha.$$

Clearly, every point in the set $\mathbb{C} = \{x \mid X = \alpha\}$ is a GNE of this game. Does a subset of these characterize economically justifiable strategic behavior? \square

Another shortcoming of the GNE is that there are settings for which not every GNE is meaningful from a real-world standpoint. The **first** motivation for our study is to present a refinement of the GNE that will retain a set of GNEs that is smaller, yet economically meaningful, even under these settings. We argue below that it is indeed the VE that has this property. Consider a game like in the above example and suppose that the Lagrange multipliers can be interpreted as *prices* charged on the players by an administrator for whom the players are anonymous. The VE is also known to be the GNE with the same Lagrange multipliers corresponding to the shared constraint [FFP07]. Thus for this game VE has the additional property of being an equilibrium with *uniform prices* whereas the GNE corresponds to one with discriminatory prices. Since players are anonymous, and hence indistinguishable from each other, it is unreasonable to assume that the administrator can charge discriminatory prices and the only equilibria that make sense are ones in which the same price is charged to all players, i.e. the VE. If the VE is indeed a refinement of the GNE, then the VE exists whenever the GNE does and thus may be used as a solution concept in lieu of the GNE.

Our **second** motivation arises from the need to characterize and compute GNEs. Consider a game in which the i^{th} player solves the parameterized convex program

$A_i(x^{-i})$	$\begin{aligned} & \underset{x_i}{\text{minimize}} && \varphi_i(x_i; x^{-i}) \\ & \text{subject to} && Ax \geq b \quad (\lambda_i), \\ & && x \geq 0, \end{aligned}$
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where $\varphi_i(x_i; x^{-i})$ is convex in x_i for all x^{-i} and λ_i is the player-specific Lagrange multiplier corresponding to the shared-constraint. Then the equilibrium conditions of the game are given by the

rank-deficient complementarity problem

$$\begin{aligned} 0 \leq x_i \perp \nabla_{x_i} f_i - A_i^T \lambda_i &\geq 0, & \forall i \in \mathcal{N}, \\ 0 \leq \lambda_i \perp Ax - b &\geq 0, & \forall i \in \mathcal{N}, \end{aligned}$$

suggesting that the original equilibrium problem is ill-posed. A simple step for making the game well-posed requires that the players have consistent multipliers, denoted by say λ , and the resulting equilibrium conditions are given by the following square complementarity problem:

$$\begin{aligned} 0 \leq x_i \perp \nabla_{x_i} f_i - A_i^T \lambda &\geq 0, & \forall i \in \mathcal{N}, \\ 0 \leq \lambda \perp Ax &\geq b, & \forall i \in \mathcal{N}. \end{aligned}$$

In general, obtaining a GNE requires a solution of this ill-posed system which leads to a *quasi-variational inequality* in the primal-space and a non-square complementarity problem in the primal-dual space. The VE, on the other hand, requires the solution of either a variational inequality (primal space) or a square complementarity problem (primal-dual space) both of which being far more tractable objects.

This has several implications both from an analytical and a computational standpoint. In particular, in attempting to analyzing GNEs, one may choose to focus primarily on VEs. Such a direction is sensible only when it is known that the the existence of a GNE is sufficient for the existence of a VE. Furthermore, as regards computation, it has been common practice [FK07, PF05, LM05] to compute the VE instead of the GNE and limit computation only to the class of games, \mathcal{S}_2 , for which the VE exists. Notice that \mathcal{S}_2 is contained in the class of games, \mathcal{S} , for which the VE is a refinement of the GNE. But sufficiency conditions for the solvability of variational inequalities (such as those in [FP03]) that may be applied for checking if a game belongs to \mathcal{S}_2 do not exploit the existence of a GNE to show a solution to $\text{VI}(\mathbb{C}, F)$; these theorems apply to a class, \mathcal{S}'_2 , smaller than \mathcal{S}_2 , for which one can claim the existence of a VE *independently* of knowledge of the existence of a GNE. This in turn has limited the practice of computation of GNEs only to games in \mathcal{S}'_2 . Identification of \mathcal{S} leads to an identification of \mathcal{S}_2 , rather than the smaller set \mathcal{S}'_2 .

The contribution of this chapter is a theory that gives sufficient conditions for a game to belong

2.1 Introduction

to \mathcal{S} . For certain classes of games, these conditions are also seen to be necessary. Also, \mathcal{S} is shown to contain classes other than \mathcal{S}'_2 in itself. Ours is perhaps the first work on the refinement of equilibria in the context of generalized Nash games. Our sufficient conditions are expressed in terms of the *Brouwer degree*, which is seen to relate the GNE and the VE in a profound manner.

In both, primal and primal-dual space, we show that there are functions v and g whose zeros are VEs and GNEs respectively, such that the Brouwer degrees of v and g , with respect to zero are equal. This paves the way for identifying subclasses of \mathcal{S} . In the primal setting, we show the above result with v and g taken as the natural maps of the quasi-variational inequality (whose solutions capture all GNEs) and the variational inequality (whose solutions are the VEs). Through a novel equation reformulation of the primal-dual GNE, this degree theoretic approach is extended to the primal-dual space. Finally, we show that these sufficiency conditions can be applied on an instance of a shared-constraint Nash-Cournot game arising in power markets.

The chapter is organized as follows. Sections 2.2 and 2.3 deal with the treatment of the refinement question in primal and primal-dual spaces, respectively. Section 2.4 contains some examples of shared-constraint games and describes how the sufficiency conditions for a game to admit the VE as a refinement may be applied to shared-constraint Nash-Cournot game arising in power markets. We conclude with some final considerations in Section 2.5. Before proceeding, we outline our assumptions and provide some technical background.

■ 2.1.1 Background

We make the following assumptions throughout the chapter.

Assumption 2.1 *For each $i \in \mathcal{N}$, the objective function $\varphi_i \in C^2$ and $\varphi_i(x_i; x^{-i})$ is convex in x_i for all x^{-i} . Unless otherwise mentioned, \mathbb{C} is closed, convex and has a nonempty interior.*

Under Assumption 2.1, x_i is optimal for $A_i(x^{-i})$ if and only if $\nabla_i \varphi_i(x)^T(y_i - x_i) \geq 0$, for all $y_i \in K_i(x^{-i})$, where K_i is as defined in (2.1). Thus if $x = (x_1, \dots, x_N)$ is a GNE of \mathcal{G} if and only if it solves the *quasi-variational inequality* (QVI) [FP03] below.

Find $x \in K(x)$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in K(x).$ (QVI(K, F))
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A quasi-variational inequality is a generalization of a variational inequality (VI). For the closed

Chapter 2. Refinement of the Generalized Nash Equilibrium

convex set \mathbb{C} and function F , the variational inequality $\text{VI}(\mathbb{C}, F)$ is the following problem, a solution of which was defined to be the VE in Definition 2.2.

Find $x \in \mathbb{C}$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in \mathbb{C}. \quad (\text{VI}(\mathbb{C}, F))$

We use $\mathbf{F}_{\mathbb{C}}^{\text{nat}} : \mathbb{R}^m \rightarrow \mathbb{R}^m$, and $\tilde{\mathbf{F}}_K^{\text{nat}} : \text{dom}(K) \rightarrow \mathbb{R}^m$ to denote the natural map of $\text{VI}(\mathbb{C}, F)$ and $\text{QVI}(K, F)$ respectively (cf. Appendix A.3).

A brief background on Brouwer degree theory follows. The Brouwer degree [Kes04, OCC06, FG95] of a function is a topological concept that allows us to claim the existence of zeros of the function in a specified open set. Degree theory has been previously applied to the study of variational inequalities [Gow93, GP94, FP03]. Let $\Omega \subset \mathbb{R}^m$ be an open bounded set, $f : \bar{\Omega} \rightarrow \mathbb{R}^m$ be continuous and $p \in \mathbb{R}^m \setminus f(\partial\Omega)$. We say the Brouwer degree of f with respect to p on Ω , denoted as $\deg(f, \Omega, p)$, is *well defined* if $p \notin f(\partial\Omega)$ and it exists only for such p . Let $\mathbf{I} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ denote the identity map. $\deg(f, \Omega, p)$ is an integer with the following properties.

1. (*Normalization*) $\deg(\mathbf{I}, \Omega, p) = 1$ if and only if $p \in \Omega$.
2. (*Solvability*) $\deg(f, \Omega, p) \neq 0$ then $f(x) = p$ for some $x \in \Omega$.
3. (*Homotopy invariance*) $\deg(H(\cdot, t), \Omega, p)$ is independent of $t \in [0, 1]$ for any continuous function $H : \bar{\Omega} \times [0, 1] \rightarrow \mathbb{R}^m$ and $p \in \mathbb{R}^m$ such that

$$p \notin \cup_{t \in [0, 1]} H(\partial\Omega, t).$$

H is called a homotopy.

4. (*Translation invariance*) $\deg(f - p, \Omega, 0) = \deg(f, \Omega, p)$
5. (*Degree of injective maps*) Let f be continuous and injective and $f(x) = p$ for some $x \in \Omega$. Then $\deg(f, \Omega, p) = \pm 1$.

Note that the converse of property 2 is *not* true in general. i.e. if $f(x) = p$ for some x in Ω and $\deg(f, \Omega, p)$ is well defined, it does not imply that the degree not zero. But if f is continuous and injective, such a claim can be made, cf. property 5.

■ 2.2 Primal generalized Nash and variational equilibria

In this section we begin the development of our theory of the refinement of the GNE. Our analysis is restricted to the primal space and does not impose any algebraic form on \mathbb{C} , relying mainly on geometric and convex analytic arguments. A *primal-dual* analysis that uses an algebraic form for \mathbb{C} and a primal-dual characterization of the GNE and VE is included in Section 2.3. The material pertaining specifically to the refinement of the GNE is encompassed in Section 2.2.2. The following material upto the end of Section 2.2.1 establishes some preliminary results useful for the results in Section 2.2.2 and for a better understanding of $\text{QVI}(K, F)$. We begin by recalling that every VE is a GNE, a result shown by Facchinei et al. [FFP07, Theorem 2.1].

Theorem 2.1 *For any continuous function $G : \mathbb{R}^m \rightarrow \mathbb{R}^m$, if x is a solution of $\text{VI}(\mathbb{C}, G)$ then x is a solution of $\text{QVI}(K, G)$.*

The next set of results help develop a deeper understanding of the set-valued map K .

■ 2.2.1 The properties of K

Fig 2.1(a) shows a convex set \mathbb{C} and $K(x)$ for an $x \in \mathbb{R}^2$, assuming $m_1 = m_2 = 1$ and $N = 2$. Notice that $K(x)$ is formed as a product, namely $K_1(x_2) \times K_2(x_1)$. In general $\text{dom}(K) := \{x \mid K(x) \neq \emptyset\}$ is not \mathbb{R}^m and there may be points outside \mathbb{C} whose image under at least one of the K_i 's is empty. For instance in Fig 2.1(a), notice the point $y = (y_1, y_2)$ for which both $K_1(y_2)$ and $K_2(y_1)$ are empty. The following Lemma mentions some more relationships between K and \mathbb{C} . See Section 2.6.3 for proof.

Lemma 2.2 *Let \mathbb{C} be a closed set in \mathbb{R}^m and K be as given in (2.1). Then the following hold:*

1. *If $\mathbb{C} = \prod_{i \in \mathcal{N}} \mathbb{C}_i$, where $\mathbb{C}_i \subseteq \mathbb{R}^{m_i}$ for every $i \in \mathcal{N}$, are nonempty, not necessarily convex sets, then $K(x) = \mathbb{C}$ for every x in \mathbb{C} and is empty otherwise.*
2. *For any \mathbb{C} , not necessarily convex, x is a fixed point of K if and only if $x \in \mathbb{C}$.*
3. *If \mathbb{C} is closed and convex, $K(x)$ is closed and convex for any $x \in \text{dom}(K)$.*

4. Let \mathbb{C} be closed and convex and $x \in \mathbb{C}$. For this x , let $K(x)_\infty$ and \mathbb{C}_∞ denote the recession cone (see Appendix A.1) of the sets $K(x)$ and \mathbb{C} , respectively. Then we have $K(x)_\infty \subseteq \mathbb{C}_\infty$. Consequently, if \mathbb{C} is bounded, $K(x)$ is bounded for every x in \mathbb{C} .

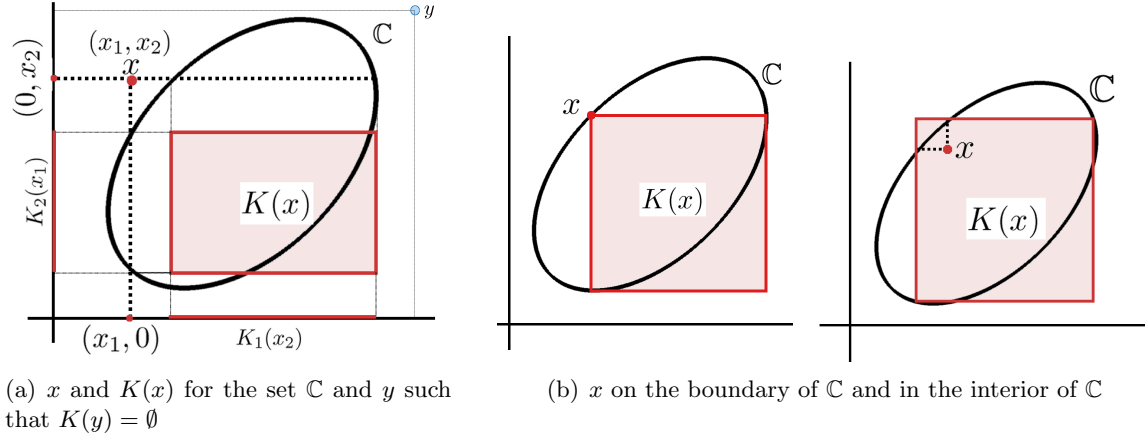


Figure 2.1: K for a set \mathbb{C} in \mathbb{R}^2

As a consequence of Lemma 2.2(2), the set of fixed points of K is nonempty when \mathbb{C} is nonempty and $\text{QVI}(K, F)$ which seeks such a fixed point as a solution is not vacuous for any such \mathbb{C} . Lemma 2.2(2) can be strengthened significantly: fixed points of K are in the interior of \mathbb{C} if and only if they are in the interior of their image under K . This is illustrated below in Fig 2.1(b) and proved in the following result. See Section 2.6.4 for proof. The notation $\text{int}(\bullet)$ and $\partial\bullet$ stand for the interior and the boundary of ‘ \bullet ’ respectively.

Lemma 2.3 *A point x belongs to the interior of $K(x)$ if and only if x is in the interior of \mathbb{C} .*

This concludes the preliminaries for this section. The following section addresses the issue of the refinement of the GNE.

■ 2.2.2 Refinement of the generalized Nash equilibrium

We begin with the formal definition of the refinement.

Definition 2.3 (Refinement) *Let \mathcal{S} be a class of shared-constraint games defined by the collection of players, their strategy spaces, their objective functions and the shared constraints. Let $\mathcal{E}(G)$*

2.2 Primal generalized Nash and variational equilibria

denote the set of GNEs of a game $G \in \mathcal{S}$ and let \mathcal{G} and \mathcal{U} be defined as

$$\mathcal{G} := \{\mathcal{E}(G) \mid G \in \mathcal{S}\} \quad \text{and} \quad \mathcal{U} := \bigcup_{G \in \mathcal{S}} \mathcal{E}(G),$$

respectively. A refinement of the GNE of games in \mathcal{S} is a set-valued mapping $\mathcal{R} : \mathcal{G} \rightarrow 2^{\mathcal{U}}$ that satisfies the following properties:

(R1) The refinement of a game must be a subset of equilibria of the game. i.e.

$$\mathcal{R}(\mathcal{E}(G)) \subseteq \mathcal{E}(G) \quad \forall G \in \mathcal{S}.$$

(R2) Any game with nonempty set of GNEs must admit a refinement. Specifically for all $G \in \mathcal{S}$,

$$\text{if } \mathcal{E}(G) \neq \emptyset, \quad \text{then } \mathcal{R}(\mathcal{E}(G)) \neq \emptyset.$$

In studying the VE as a refinement of the GNE, we consider the following rule for generating a refinement of the game \mathcal{G} :

$$\mathcal{R}(\mathcal{E}(\mathcal{G})) = \mathcal{R}(\text{SOL}(\text{QVI}(K, F))) := \text{SOL}(\text{VI}(\mathbb{C}, F)). \quad (2.2)$$

Let \mathcal{S}_1 be the class of shared-constraint games that admit a GNE and let \mathcal{S}_2 be those that admit a VE. By Theorem 2.1, we know that $\mathcal{S}_2 \subseteq \mathcal{S}_1$, so \mathcal{R} as defined in (2.2), satisfies **(R1)**. To confirm the VE as a refinement for games in \mathcal{S} we need that \mathcal{R} satisfies **(R2)**, which, for the game \mathcal{G} from Section 2.1 amounts to showing

$$\text{SOL}(\text{QVI}(K, F)) \neq \emptyset \quad \implies \quad \text{SOL}(\text{VI}(\mathbb{C}, F)) \neq \emptyset. \quad (2.3)$$

If $\widetilde{\mathcal{S}}_1$ is the class of games for which a GNE does not exist, the class for which \mathcal{R} is a refinement is given by $\mathcal{S}_2 \cup \widetilde{\mathcal{S}}_1 := \mathcal{S}$.

A natural question one may ask is whether \mathcal{S} is the class of all games. This is answered in the negative by the following counter-example of a game with a (unique) GNE but no VE.

Example 2.2. Game with unique GNE and no VE: Let $\mathbb{C} = \{(x_1, x_2) \mid x_2 \geq e^{-x_1}, x_1 \geq 0\}$, and

$$K(x) = \{(y_1, y_2) \mid y_2 \geq e^{-x_1}, x_1 \geq 0, y_1 \geq 0, x_2 \geq e^{-y_1}\}.$$

Let $F(x) = (1 + x_1 - \frac{1}{x_2}, 1)$. It is easily verified that $x = (0, 1)$ satisfies

$$\left(1 + x_1 - \frac{1}{x_2}\right)(y_1 - x_1) + (y_2 - x_2) \geq 0 \quad \forall y \in K(x), \quad (2.4)$$

and thus $(0, 1)$ is a GNE. To show the uniqueness of this GNE, we assume a GNE $x \neq (0, 1)$ exists in \mathbb{C} and arrive at a contradiction. For such an x , we must have $x_2 \geq e^{-x_1}$, but for this x to be a solution we note that it must satisfy $x_2 = e^{-x_1}$. This follows from the observation that the points $\{(x_1, y_2) \mid y_2 \in [e^{-x_1}, \infty)\}$ lie in $K(x)$, so if $x_2 > e^{-x_1}$, then the point $y = (x_1, e^{-x_1}) \in K(x)$ will not satisfy the QVI condition (2.4) and thus x cannot solve the QVI. Now since $x_2 = e^{-x_1}$, the point $(y_1, x_2) = (y_1, e^{-x_1})$ lies in $K(x)$ for all $y_1 \in [x_1, \infty)$. If x is a solution of the QVI, for such points we require

$$\left(1 + x_1 - \frac{1}{x_2}\right)(y_1 - x_1) \geq 0 \quad \forall y_1 \in [x_1, \infty).$$

The term in the first bracket is strictly negative since $x_2 = e^{-x_1}$ and $x \neq (0, 1)$, while the term in the second bracket can be made positive for $y > x_1$. Thus x cannot be a solution and $(0, 1)$ is the only solution. Since every VE is a GNE, this game can have at most one VE, i.e. $(0, 1)$. But for $(0, 1)$ to be a VE we require

$$(0, 1)^T \left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \geq 0 \quad \forall y \in \mathbb{C}.$$

It is easy to check that $y = (2, e^{-2}) \in \mathbb{C}$ and does not satisfy this. Thus this game has no VE but a unique GNE. \square

In effect \mathcal{S} is smaller than the class of *all* shared-constraint games, and therefore our efforts in this work are focused on identifying subclasses of \mathcal{S} . A subclass of \mathcal{S}_2 is known for which a VE can be shown to exist *without* using the hypothesis that a GNE exists; we denote this class by \mathcal{S}'_2 . Examples of \mathcal{S}'_2 are games where \mathbb{C} is compact or where F is coercive (cf. (2.11)). For

2.2 Primal generalized Nash and variational equilibria

such games we have $\text{SOL}(\text{VI}(\mathbb{C}, F)) \neq \emptyset$ (and hence $\text{SOL}(\text{QVI}(K, F))$ cannot be empty) and (2.3) holds. Therefore, while identifying subclasses of \mathcal{S} , we will also be interested in whether (a) there is any class larger than \mathcal{S}'_2 included in \mathcal{S}_2 and (b) whether there is any unifying criterion that may be articulated in terms of F and \mathbb{C} that determines \mathcal{S} . (a) is answered in the affirmative in Section 2.2.3, whereas for (b) we see that the Brouwer degree holds promise, in a way made precise below. While the theory we develop focuses on sufficient conditions for a game to have membership in \mathcal{S} , for a class of games these conditions are also seen to be necessary. In this development, we ignore settings in which it is possible to claim the existence of a VE independently of the existence of a GNE (i.e. \mathcal{S}'_2) and make the VE a refinement by default. Many of these emerge as special cases of our results.

We begin by noting a simple consequence of Lemma 2.3 - in the interior of \mathbb{C} the GNE and VE are equivalent. Thus, the VE is a refinement for every \mathcal{G} that has a GNE in the interior of \mathbb{C} and this GNE is also a VE. This is established in the theorem below.

Theorem 2.4 *Let $x \in \text{int}(K(x))$. Then x is a GNE of \mathcal{G} if and only if x is a VE.*

Proof : Due to Theorem 2.1, it suffices to prove the “only if” part of the claim. Suppose $x \in \text{int}(K(x))$ is a GNE. By Lemma 2.3, $x \in \text{int}(\mathbb{C})$. It follows that one can construct a ball, $B(x, r)$, centered at x with sufficiently small radius r , such that $B(x, r)$ is contained in $K(x) \cap \mathbb{C}$. Since x is a GNE, it follows that

$$F(x)^T(y - x) \geq 0, \quad \forall y \in B(x, r). \quad (2.5)$$

Putting $y = x + re$ and $y = x - re$ for an arbitrary unit vector e gives $F(x)^T e = 0$. Since this holds for each unit vector e , we must have $F(x) = 0$. As a consequence, x solves $\text{VI}(\mathbb{C}, F)$. ■

Theorem 2.4 should not be surprising. If \mathbb{C} is specified using a continuously differentiable algebraic constraint $c(\cdot) \geq 0$, the hypothesis $x \in \text{int}(K(x))$ reduces to $c(x) > 0$. If x is a GNE, the Lagrange multipliers corresponding to $c(\cdot) > 0$ would be zero. x is an equilibrium with *shared* ($= 0$) multiplier and therefore a VE (see Theorem 2.12 in Section 2.3).

Recall the definitions of $\tilde{\mathbf{F}}_K^{\text{nat}}$ and $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$, the natural maps of $\text{QVI}(K, F)$ and $\text{VI}(\mathbb{C}, F)$ from Section 2.1.1. The result that our theory is built on is Theorem 2.5. We show that the Brouwer degrees of $\tilde{\mathbf{F}}_K^{\text{nat}}$ and $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$ with respect to zero, whenever well defined, are equal. Recall from Section

Chapter 2. Refinement of the Generalized Nash Equilibrium

2.1.1 that if K is continuous, $\tilde{\mathbf{F}}_K^{\text{nat}}$ is continuous.

Theorem 2.5 *Let Ω be an open bounded set such that $\bar{\Omega} \subseteq \text{dom}(K)$ and suppose K is continuous on $\bar{\Omega}$. If $0 \notin \tilde{\mathbf{F}}_K^{\text{nat}}(\partial\Omega)$, then*

$$\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0) = \deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0).$$

Proof : First observe that because every VE is a GNE, the assumption that $0 \notin \tilde{\mathbf{F}}_K^{\text{nat}}(\partial\Omega)$ implies that $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$ is not zero on $\partial\Omega$. Thus $\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0)$ and $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0)$ are both well defined.

We will use the invariance of the Brouwer degree under homotopy (property 3 of from Section 2.1.1) to prove the claim. Define $H : [0, 1] \times \text{dom}(K) \rightarrow \mathbb{R}^m$ as

$$H(\bar{t}, v) = \bar{t}\mathbf{F}_{\mathbb{C}}^{\text{nat}}(v) + (1 - \bar{t})\tilde{\mathbf{F}}_K^{\text{nat}}(v) \quad \forall \bar{t} \in [0, 1], v \in \bar{\Omega}.$$

By continuity of K , H is a homotopy between $\tilde{\mathbf{F}}_K^{\text{nat}}$ and $\mathbf{F}_{\mathbb{C}}^{\text{nat}}$. By property 3 of the Brouwer degree, if $0 \notin \bigcup_{t \in [0, 1]} H(t, \partial\Omega)$, we would have $\deg(H(1, \cdot), \Omega, 0) = \deg(H(0, \cdot), \Omega, 0)$, by which the required result would follow.

We have already seen $0 \notin H(1, \partial\Omega) \cup H(0, \partial\Omega)$. So it suffices that $0 \notin H(\bar{t}, \partial\Omega)$ for all $\bar{t} \in (0, 1)$ for the result to follow. Assume that this is not so. i.e. assume that for some $t \in (0, 1)$ and $z \in \partial\Omega$, $H(t, z) = 0$. Then

$$z = tx^c + (1 - t)x^k,$$

where $x^k = \Pi_{K(z)}(z - F(z))$ and $x^c = \Pi_{\mathbb{C}}(z - F(z))$. Since $x^k \in K(z)$, $(x_i^k, z^{-i}) \in \mathbb{C}$ for every $i \in \mathcal{N}$, implying that the point x^a belongs to \mathbb{C} , where

$$x^a := \frac{1}{N} \sum_{i \in \mathcal{N}} (x_i^k, z^{-i}) = \frac{(N-1)}{N} z + \frac{1}{N} x^k.$$

Indeed, one may verify that

$$z = \frac{N(1-t)}{N(1-t)+t} x^a + \frac{t}{N(1-t)+t} x^c,$$

2.2 Primal generalized Nash and variational equilibria

implying that z is also in \mathbb{C} , or equivalently in $K(z)$. Now using the property of projection in Lemma A.1, we get

$$(y - x^c)^T(x^c - z + F(z)) \geq 0 \quad \forall y \in \mathbb{C} \quad \text{and} \quad (y - x^k)^T(x^k - z + F(z)) \geq 0, \quad \forall y \in K(z)$$

Since $z \in K(z) \cap \mathbb{C}$, we may put $y = z$ in both of the above inequalities to get

$$F(z)^T(z - x^c) \geq \|z - x^c\|^2 \geq 0 \quad \text{and} \quad F(z)^T(z - x^k) \geq \|z - x^k\|^2 \geq 0. \quad (2.6)$$

On the other hand since $z - x^c = -\frac{1-t}{t}(z - x^k)$, we have

$$-\frac{1-t}{t}F(z)^T(z - x^k) \geq 0,$$

which from (2.6) gives $F(z)^T(z - x^k) = 0$ and $z = x^k$. But this means that $\tilde{\mathbf{F}}_K^{\text{nat}}(z) = 0$, a contradiction to the hypothesis that $0 \notin \tilde{\mathbf{F}}_K^{\text{nat}}(\partial\Omega)$. Hence $\deg(H(t, \cdot), \Omega, 0)$ is well defined for all $t \in [0, 1]$. By property 3 of the Brouwer degree, its value is independent of t , whence the result follows. ■

The above result is of a deeper flavor than Theorem 2.1 of Facchinei et al., for it shows a symmetric relationship (equality, rather than a one-way inclusion) between the GNE and the VE. Indeed, it says that the GNE and the VE are equivalent upto the degree of the corresponding natural maps. Moreover, the only assumptions the theorem makes are those necessary for these degrees to be well defined and the result may thereby be thought of as being germane to such games.

Theorem 2.5 and the solvability property of the Brouwer degree allow for concluding the validity of the implication in (2.3) through the degree of $\tilde{\mathbf{F}}_K^{\text{nat}}$. Specifically, if the nonemptiness of $\text{SOL}(\text{QVI}(K, F))$ implies the nonzeroness of $\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0)$ for some Ω , i.e. the *converse* of property 2 of the Brouwer degree holds, then it also implies the nonemptiness of $\text{SOL}(\text{VI}(\mathbb{C}, F))$ and the game admits the VE as a refinement of the GNE. This is articulated in the following theorem.

Theorem 2.6 *Consider game \mathcal{G} and suppose K is continuous. Consider the following statements:*

(C0) \mathcal{G} admits a GNE.

(C1) *There exists an open bounded set, Ω , with $\bar{\Omega} \subseteq \text{dom}(K)$ such that Ω contains a GNE of \mathcal{G} ,*

Chapter 2. Refinement of the Generalized Nash Equilibrium

and has no GNE of \mathcal{G} on its boundary.

(C2) For an open bounded set, Ω with $\bar{\Omega} \subseteq \text{dom}(K)$, $\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0)$ is well defined and nonzero.

(C3) \mathcal{G} admits a VE.

Then,

- (a) we have (C2) \implies (C3). Consequently if (C2) holds for \mathcal{G} , then \mathcal{G} admits a GNE and a VE. \mathcal{G} belongs to the aforementioned class \mathcal{S} and the implication in (2.3) holds for \mathcal{G} .
- (b) If \mathcal{G} has the property that

$$(\mathbf{C0}) \implies (\mathbf{C1}) \implies (\mathbf{C2}), \quad \text{or} \quad (\mathbf{C0}) \implies (\mathbf{C2})$$

then \mathcal{G} belongs to \mathcal{S} and (2.3) holds.

Proof :

- (a) Assume (C2) holds. Then by Theorem 2.5 we get $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) \neq 0$. By solvability property 2 of the Brouwer degree we conclude that there exists $x \in \Omega$ such that $\mathbf{F}_{\mathbb{C}}^{\text{nat}}(x) = 0$. But this means that x solves $\text{VI}(\mathbb{C}, F)$ and is hence a VE. It follows that if (C2) holds for \mathcal{G} then \mathcal{G} admits a GNE and VE and the implication in (2.3) holds.
- (b) If (C0) \implies (C1) \implies (C2) or if (C0) \implies (C2), then we may say that (C0), i.e. existence of a solution to $\text{QVI}(K, F)$, is sufficient for $\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0)$ to be nonzero. Then using part (a), we conclude (C0) implies that \mathcal{G} admits a VE. Thus, for \mathcal{G} , the existence of a GNE is a sufficient condition for the existence of a VE and (2.3) holds. ■

The next theorem shows that for a class of games, (a) and (b) are not just sufficient but also necessary for (2.3) to hold. This indicates that using Theorem 2.6 to claim (2.3) is an apt approach.

Theorem 2.7 *Let K be continuous, $\text{SOL}(\text{QVI}(K, F))$ be bounded and Ω be an open bounded set with $\bar{\Omega} \subseteq \text{dom}(K)$ containing $\text{SOL}(\text{QVI}(K, F))$. If F is pseudo-monotone, the implication*

$$\text{SOL}(\text{QVI}(K, F)) \neq \emptyset \implies \text{SOL}(\text{VI}(\mathbb{C}, F)) \neq \emptyset \tag{2.7}$$

2.2 Primal generalized Nash and variational equilibria

holds if and only if the following implication holds.

$$\text{SOL}(\text{QVI}(K, F)) \neq \emptyset \implies \deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0) \neq 0. \quad (2.8)$$

Proof : We first recall from Theorem 2.3.17 in [FP03] that since the solution set of $\text{VI}(\mathbb{C}, F)$ (if nonempty) is bounded and F is pseudo-monotone,

$$\text{SOL}(\text{VI}(\mathbb{C}, F)) \neq \emptyset \iff \deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) \neq 0. \quad (2.9)$$

The proof of the result is now easy to see. Suppose (2.7) holds. Then combining (2.7), (2.9) and Theorem 2.5, we get (2.8). Conversely, if (2.8) holds, then using Theorem 2.5 we get

$$\text{SOL}(\text{QVI}(K, F)) \cap \Omega \neq \emptyset \implies \deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) \neq 0,$$

which by the solvability property of the Brouwer degree leads to (2.7). ■

Theorem 2.6 forms the basis for identifying subclasses of \mathcal{S} . It provides a framework, via (C1) and (C2), for identifying games for which the implication (C0) \implies (C3) holds. More complex chains of implications than those in (a) and (b) may also be framed using (C0) – (C3) for this purpose. Note that (C2) does not follow from (C1), since in general the converse of solvability property 2 of the Brouwer degree does not hold.

We now apply Theorem 2.6 to identify subclasses of \mathcal{S} . A sufficient condition for \mathcal{G} to satisfy, “(C0) \implies (C1) \implies (C2)” is that $\tilde{\mathbf{F}}_K^{\text{nat}}$ be continuous and one-to-one. But since generalized Nash games are known to have manifolds of equilibria, we expect $\tilde{\mathbf{F}}_K^{\text{nat}}$ to have manifolds of zeros in fairly general cases (see Theorem 2.17). Consequently, it is unlikely that $\tilde{\mathbf{F}}_K^{\text{nat}}$ can be shown to be one-to-one in general and we do not attempt that line of research. Instead we observe that in (C2) (and in (2.8)), we may ask that $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0)$ be nonzero, thanks to Theorem 2.1, and use this to identify these subclasses. We begin by showing that certain classes in \mathcal{S}'_2 are identified by Theorem 2.6 (where \mathcal{S}'_2 denotes the subset of \mathcal{S}_2 for which the existence of the VE can be shown without the knowledge of the existence of a GNE). In particular, we see that Theorem 2.6 includes in it the well known fact: if \mathbb{C} is compact, the VE and GNE both exist and hence the VE is a

Chapter 2. Refinement of the Generalized Nash Equilibrium

refinement. Next we show Theorem 2.6 contains games where F is a coercive mapping as a special case. In both of these proofs we will consider an open bounded set $\Omega \subseteq \mathbb{R}^m$ and consider the homotopy $H : [0, 1] \times \overline{\Omega} \rightarrow \mathbb{R}^m$ given by

$$H(t, x) = x - \Pi_{\mathbb{C}}(t(x - F(x)) + (1 - t)x^{\text{ref}}) \quad \forall x \in \overline{\Omega}, t \in [0, 1], \quad (2.10)$$

for a specific choice of $x^{\text{ref}} \in \Omega \cap \mathbb{C}$. The hypotheses of the particular result are then shown to imply that $H(1, \cdot) = \mathbf{F}_{\mathbb{C}}^{\text{nat}}$ is homotopic to $H(0, \cdot) = \mathbf{I} - x^{\text{ref}}$, which by property 3 of the Brouwer degree gives $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) = 1$. Note that the existence of the GNE, i.e. (C0), is not used and (C2) is claimed directly.

Lemma 2.8 (Brouwer's fixed point theorem) *Let \mathbb{C} be compact and suppose Ω is an open bounded set large enough to strictly contain \mathbb{C} . Then $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) = 1$ and the game admits a VE and a GNE.*

Proof : Let $x^{\text{ref}} \in \mathbb{C}$ be some point and for Ω as given, consider the homotopy H in (2.10). Since Ω strictly contains \mathbb{C} , $\partial\Omega \cap \mathbb{C} = \emptyset$. Since for any t , zeros of $H(t, \cdot)$ lie in \mathbb{C} , we must have $H(t, \partial\Omega) \neq 0$ for every $t \in [0, 1]$. By property 3 of the Brouwer degree, $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) = \deg(\mathbf{I} - x^{\text{ref}}, \Omega, 0) = 1$, where the last equality is because x^{ref} belongs to $\mathbb{C} \subset \Omega$. By the solvability property 2 of the Brouwer degree, the game admits a VE. Now we may use Theorem 2.1 to conclude that the game also admits a GNE. ■

We next see that if F is coercive, then the VE is a refinement. The proof below is adapted from [FP03, Proposition 2.2.3].

Lemma 2.9 *Suppose there exists an $x^{\text{ref}} \in \mathbb{C}$ such that*

$$\liminf_{x \in \mathbb{C}, \|x\| \rightarrow \infty} F(x)^T(x - x^{\text{ref}}) > 0. \quad (2.11)$$

There exists an open bounded set Ω such that $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) = 1$ and this game has a VE and GNE.

Proof : Since the limit above is positive, there exists an open bounded set Ω such that $F(x)^T(x - x^{\text{ref}}) > 0$ for all $x \in \partial\Omega \cap \mathbb{C}$. Without loss of generality one may take Ω large enough to contain

2.2 Primal generalized Nash and variational equilibria

x^{ref} . Since $x^{\text{ref}} \in \mathbb{C}$, it follows that no point on the set $\partial\Omega \cap \mathbb{C}$ solves $\text{VI}(\mathbb{C}, F)$. Let H be as in (2.10). It is easy to see that $0 \notin H(1, \partial\Omega) \cup H(0, \partial\Omega)$. Assume that $H(t, x) = 0$ for some $t \in (0, 1)$ and $x \in \partial\Omega$. Since zeros of $H(t, \cdot)$ must lie in \mathbb{C} , $x \in \partial\Omega \cap \mathbb{C}$. Now by Lemma A.1

$$(y - x)^T(x - t(x - F(x)) - (1 - t)x^{\text{ref}}) \geq 0 \quad \forall y \in \mathbb{C},$$

whereby for $y = x^{\text{ref}}$,

$$F(x)^T(x - x^{\text{ref}}) < -\frac{1-t}{t}\|x - x^{\text{ref}}\|^2 < 0.$$

Since $x \in \partial\Omega$, this is a contradiction. Consequently, our assumption that $H(t, x) = 0$ is incorrect and we must have $0 \notin \bigcup_{t \in [0,1]} H(t, \partial\Omega)$. Therefore property 3 of the Brouwer degree $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) = \deg(\mathbf{I} - x^{\text{ref}}, \Omega, 0) = 1$, whereby this game has a VE and by Theorem 2.1, a GNE. ■

Notice that since in Lemma 2.8 and Lemma 2.9 we are essentially proving $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0)$ to be nonzero, the requirement of the continuity of K imposed in Theorem 2.5 has been relaxed and we have instead argued the existence of the GNE using Theorem 2.1 after proving the existence of a VE as above.

We now turn to an existence result for QVIs that claims (C2), and thus provides a sufficient condition for the VE to be a refinement of the GNE. This result is Corollary 2.8.4 in [FP03, page 222], which applies to *any* QVI, not necessarily those arising from K as defined as in (2.1). We have reproduced it below with the added hypothesis “ $0 \notin \tilde{\mathbf{F}}_K^{\text{nat}}(\partial\Omega)$ ”.

Theorem 2.10 (Theorem 2.8.3 and Corollary 2.8.4 [FP03]) *Let $K : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$ be a closed-valued and convex-valued point-to-set map. Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a continuous function. Suppose there exists an open bounded set $\Omega \subset \text{dom}(K)$ and a vector $x^{\text{ref}} \in \Omega$ such that $0 \notin \tilde{\mathbf{F}}_K^{\text{nat}}(\partial\Omega)$ and*

1. K is continuous on $\overline{\Omega}$
2. x^{ref} belongs to $K(x)$ for every $x \in \overline{\Omega}$
3. the following holds

$$\{x \in K(x) : (x - x^{\text{ref}})^T F(x) < 0\} \cap \partial\Omega = \emptyset.$$

Then $\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0) \neq 0$.

Chapter 2. Refinement of the Generalized Nash Equilibrium

For a shared-constraint game such as \mathcal{G} , condition (3) in Theorem 2.10 above is equivalent to requiring that

$$\{x \in \mathbb{C} : (x - x^{\text{ref}})^T F(x) < 0\} \cap \partial\Omega = \emptyset,$$

since by using Lemma 2.2(2), $x \in K(x) \iff x \in \mathbb{C}$. This in turn is known to be a sufficient condition for $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0)$ to be well defined and nonzero, by Lemma 2.9. In this way, Theorem 2.5 leads to an alternative proof for the above theorem for K as in (2.1) and unifies Proposition 2.2.3 and Corollary 2.8.4 in [FP03] which are the chief existence results for VIs and QVIs respectively.

■ 2.2.3 Identification of subclasses of $\mathcal{S} \setminus \mathcal{S}'_2$

In section we concentrate on the identification of classes of games in \mathcal{S}_2 that do not lie in \mathcal{S}'_2 . We show by an argument based on Theorem 2.6, that the class of games with pseudo-monotone F and with certain other properties of the recession cone of \mathbb{C} have the VE as a refinement. Recall that in the proof of Theorem 2.7 we had seen that if F is pseudo-monotone and $\text{SOL}(\text{VI}(\mathbb{C}, F))$ is nonempty and bounded then $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0)$ is well defined and nonzero over any neighbourhood Ω containing $\text{SOL}(\text{VI}(\mathbb{C}, F))$. Our next result is an “extension” of this fact to $\text{QVI}(K, F)$: for a certain class of games **(C0)** implies that either **(C3)** holds or **(C2)** holds, and as a consequence **(C0) \implies (C3)**. Definitions of the normal cone (\mathcal{N}), tangent cone (\mathcal{T}), recession cone (\mathbb{C}_{∞}) and dual cone (denoted by S^* for a cone S) used below can be found in Appendix A.1.

Theorem 2.11 *Suppose F is pseudo-monotone and x^{ref} is a GNE of \mathcal{G} . Consider the following conditions:*

- (1) *Either $F(x^{\text{ref}}) = 0$ or $-F(x^{\text{ref}}) \in \text{int}(\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}})))$,*
- (1') *$\mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))^* \setminus \{0\} \subseteq \text{int}(\mathbb{C}_{\infty}^*)$,*
- (2) *$\mathbb{C}_{\infty} \subseteq \mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))$.*

If \mathcal{G} has the property that condition (2) holds and either (1) or (1') holds then \mathcal{G} admits a VE.

Proof : Our result is proved through the following set of steps.

Step 1: If \mathbb{C} is bounded there is nothing to prove, cf. Lemma 2.8, so we assume that \mathbb{C} is unbounded and prove the result by contradiction. Suppose \mathcal{G} has no VE. Therefore for any open bounded set

2.2 Primal generalized Nash and variational equilibria

Ω containing x^{ref} , $0 \notin \mathbf{F}_{\mathbb{C}}^{\text{nat}}(\partial\Omega)$. Fix such an Ω and consider the homotopy H from (2.10). Since $x^{\text{ref}} \in \Omega$, we have $0 \notin H(0, \partial\Omega) \cup H(1, \partial\Omega)$. If we have $0 \notin \bigcup_{t \in (0,1)} H(t, \partial\Omega)$, then we would get $\deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0) = \deg(\mathbf{I} - x^{\text{ref}}, \Omega, 0) = 1$, implying that \mathcal{G} has a VE in Ω . This contradicts our assumption. So we must have $H(t, x) = 0$ for some $x \in \partial\Omega \cap \mathbb{C}$ and $t \in (0, 1)$. i.e. for such x, t we have $x = \Pi_{\mathbb{C}}(t(x - F(x)) + (1 - t)x^{\text{ref}})$. Therefore by Lemma A.1,

$$(y - x)^T(x - t(x - F(x)) - (1 - t)x^{\text{ref}}) \geq 0 \quad \forall y \in \mathbb{C}.$$

As in Lemma 2.9, putting $y = x^{\text{ref}}$ gives

$$F(x)^T(x - x^{\text{ref}}) < -\frac{1-t}{t}\|x - x^{\text{ref}}\|^2 < 0.$$

Now since F is pseudo-monotone, it follows that $F(x^{\text{ref}})^T(x - x^{\text{ref}}) \leq 0$. Since Ω was arbitrary, we conclude that for each open bounded set Ω containing x^{ref} , there exists an $x \in \partial\Omega \cap \mathbb{C}$ such that $F(x^{\text{ref}})^T(x - x^{\text{ref}}) \leq 0$.

Step 2: Let $\{\Omega_k\}$ be a sequence of increasing open balls, each containing x^{ref} , such that $\bigcup_{k \in \mathbb{N}} \Omega_k = \mathbb{R}^m$. Let $x_k \in \partial\Omega_k \cap \mathbb{C}$ be such that $F(x^{\text{ref}})^T(x_k - x^{\text{ref}}) \leq 0$. Assume, without loss of generality, that the sequence $\left\{ \frac{x_k - x^{\text{ref}}}{\|x_k - x^{\text{ref}}\|} \right\}$ is convergent and let its limit be d' . We have $d' \neq 0$ and

$$F(x^{\text{ref}})^T d' \leq 0. \tag{2.12}$$

Step 3: Next, we prove that $d' \in \mathbb{C}_{\infty}$. Let $\tau \geq 0$ be arbitrary. Since $\|x_k\| \rightarrow \infty$, for sufficiently large k ,

$$\frac{\tau}{\|x_k - x^{\text{ref}}\|} \in [0, 1], \quad \text{whereby} \quad u_k := x^{\text{ref}} + \tau \frac{x_k - x^{\text{ref}}}{\|x_k - x^{\text{ref}}\|} \in \mathbb{C}.$$

By closedness of \mathbb{C} , $\lim_{k \rightarrow \infty} u_k = x^{\text{ref}} + \tau d' \in \mathbb{C}$. Since τ is arbitrary and \mathbb{C} is convex, d' is a recession direction of \mathbb{C} .

Step 4: To finish the proof, recall that the normal cone and the tangent cone of a convex set are related in the following way [Roc97, FP03]

$$-\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}})) = \mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))^*.$$

Chapter 2. Refinement of the Generalized Nash Equilibrium

Furthermore, since x^{ref} is a GNE of \mathcal{G} , $F(x^{\text{ref}}) \in \mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))^*$. Suppose (1) holds. If $F(x^{\text{ref}}) = 0$ then x^{ref} itself is a VE and there is nothing to prove. If $-F(x^{\text{ref}})$ lies in $\text{int}(\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}})))$ then for all nonzero vectors d in $\mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))$ we must have

$$F(x^{\text{ref}})^T d > 0. \quad (2.13)$$

But because (2) holds, (2.13) must also hold for all nonzero d in \mathbb{C}_∞ . Putting $d = d'$ in (2.13) contradicts (2.12). Now suppose condition (1') holds. Then $F(x^{\text{ref}}) \in \text{int}(\mathbb{C}_\infty^*)$ and so (2.13) is satisfied by $d = d'$; a contradiction to (2.12) is reached. Thus our initial assumption is incorrect; \mathcal{G} must admit a VE. ■

Note that the pseudo-monotonicity of F and the properties of \mathbb{C} mentioned in Theorem 2.11(1'),(2) are by themselves insufficient for the existence of a VE of \mathcal{G} . But given that a GNE x^{ref} exists, the above theorem provides sufficient conditions for \mathcal{G} to have a VE. The above theorem is thus seen to identify a class of games lying in $\mathcal{S} \setminus \mathcal{S}'_2$. It is not hard to see that (2) is satisfied in two cases in which we have already seen the VE to be a refinement: the case where \mathbb{C} is compact (in this case $\mathbb{C}_\infty = \{0\}$ and is included in *any* cone; herein (1') from Theorem 2.11 also holds) and the case where $x^{\text{ref}} \in \text{int}(K(x^{\text{ref}}))$ (here $\mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}})) = \mathbb{R}^m$ and (1) from Theorem 2.11 holds, since $F(x^{\text{ref}}) = 0$). Furthermore condition (1) is necessary, since the existence of a VE necessitates the existence of a GNE satisfying (1) (take $x^{\text{ref}} = \text{VE}$). Next, we discuss some instances where the above sufficiency condition may be applied.

Example 2.3. Generalized Nash game with affine shared constraints: Consider a game \mathcal{G} where F is pseudo-monotone and $\mathbb{C} = \{x \mid Ax \geq b, x \geq 0\}$ for some nonnegative $b \in \mathbb{R}^n$ and $n \times m$ matrix A with nonnegative elements. Let x^{ref} be a GNE such that $Ax^{\text{ref}} = b$. Suppose $A = [a_1, \dots, a_N]$, where $a_i \in \mathbb{R}^{n \times m_i}$. We have $\mathbb{C}_\infty = \{d \mid Ad \geq 0, d \geq 0\}$ implying that $\mathbb{C}_\infty^* = \{A^T \lambda \mid \lambda \geq 0\}$. Then $K(x^{\text{ref}})$ and $\mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))$ are given as

$$K(x^{\text{ref}}) = \prod_{i \in N} \{y_i \mid a_i y_i + \sum_{j \neq i} a_j x_j^{\text{ref}} \geq b, y_i \geq 0\}, \quad \mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}})) = \prod_{i \in N} \{d_i \mid a_i d_i \geq 0, d_i \geq 0\}.$$

Clearly, $\mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}})) \subseteq \mathbb{C}_\infty$. But by noting that A has nonnegative entries, we have that $\mathbb{C}_\infty = \mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))$. Therefore if (1) from Theorem 2.11 holds, we conclude from Theorem 2.11

2.2 Primal generalized Nash and variational equilibria

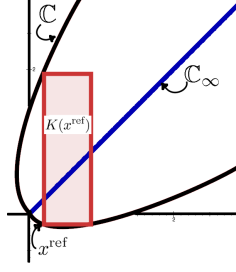


Figure 2.2: Example where Theorem 2.11(1') holds

that the game \mathcal{G} also admits a VE. \square

Example 2.4. Generalized Nash game with non-affine shared constraints: Evidently, Theorem 2.11 can apply to numerous other games with non-affine constraints since requirement (1') from Theorem 2.11 is not very restrictive but the lack of a general expression for \mathbb{C}_∞ makes it harder to provide examples. To illustrate games where (1') from Theorem 2.11 may hold, we present the following example in \mathbb{R}^2 . Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be pseudo-monotone, $\mathbb{C} = \{(x_1, x_2) \mid x_1^2 - 2x_1x_2 + x_2^2 - x_1\sqrt{2} - x_2\sqrt{2} \leq 0\}$, be the epigraph of a tilted parabola. Thus $\mathbb{C}_\infty = \{(x_1, x_2) \mid x_1 = x_2 \geq 0\}$, whereas for x^{ref} as shown in Fig 2.2, $\mathcal{J}(x^{\text{ref}}; K(x^{\text{ref}})) = [0, \infty) \times [0, \infty)$. Thus Theorem 2.11(1') holds (so Theorem 2.11(2) also holds). Hence if x^{ref} is a GNE, \mathcal{G} admits a VE. \square

Following are some concluding remarks about the above result. Theorem 2.11 can be claimed via Theorem 2.3.5, in [FP03, page 158]. We have avoided that path in order to demonstrate the reach of Theorem 2.6. The 'int' in conditions (1) and (1') presupposes that cones $\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}}))$ and \mathbb{C}_∞^* have an interior. The 'int' may be relaxed to 'relative interior' if the cones satisfy some regularity; see [FP03, ch. 2.4.1] for details. It is easy to show (see, e.g., [RW09]) that

$$\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}})) = \prod_{i \in \mathcal{N}} \mathcal{N}(x_i^{\text{ref}}; K_i(x^{\text{ref}, -i})).$$

When \mathbb{C} is given as an algebraic constraint $\{v \mid c(v) \geq 0\}$ for a continuously differentiable function $c : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and x^{ref} is on the boundary of \mathbb{C} ,

$$\mathcal{N}(x_i^{\text{ref}}; K_i(x^{\text{ref}, -i})) = \{\alpha^T \nabla_i c(x^{\text{ref}}) \mid \alpha \in \mathbb{R}^n \geq 0\}.$$

Hence $\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}}))$ is at most nN dimensional. In this setting, for $\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}}))$ to have a nonempty interior, it is necessary that $m \leq nN$. When we do have the nonemptiness of $\text{int}(\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}})))$, Theorem 2.11(1) has another interpretation. It says that either $F(x^{\text{ref}}) = 0$ or the Lagrange multipliers corresponding to the constraint c are all strictly positive at x^{ref} . Interestingly, condition (1) in Theorem 2.11 and the requirement that $m \leq nN$ also appear in the sufficient condition for the existence of a manifold of GNEs (Theorem 2.17 in Section 2.6.2). The connections between Theorems 2.11 and 2.17 are being studied further as part of ongoing research.

Review of sufficiency conditions: We now summarize the contributions of this section. The natural maps corresponding to the GNE and the VE were shown to have equal Brouwer degree (cf. Theorem 2.5). Therefore a sufficient condition for a class (denoted \mathcal{S}) to have the VE as a refinement of the GNE is that the existence of a GNE imply the nonzeroness of the degree of these natural maps (cf. Theorem 2.6). This condition is also necessary for a certain class of games (cf. Theorem 2.7). We divide \mathcal{S} into further subclasses: a class (denoted by \mathcal{S}'_2) with properties such that the existence of the VE can be claimed independently of the knowledge of the GNE; and the class where the VE exists if a GNE exists ($\mathcal{S} \setminus \mathcal{S}'_2$). Lemmas 2.8 and 2.9 showed that classes in \mathcal{S}'_2 , i.e. the class of games with compact \mathbb{C} and those with a coercive F , are special cases of Theorem 2.6 and thus admit the VE as a refinement of the GNE. Subclasses of $\mathcal{S} \setminus \mathcal{S}'_2$ were identified in Theorem 2.11 as games with a pseudo-monotone mapping F and with recession cones \mathbb{C}_∞ admitting certain properties. In summary, we have developed a host of broad verifiable conditions for claiming whether shared-constraint Nash games can admit a VE as a refinement of the GNE.

■ 2.3 Primal-dual generalized Nash and variational equilibria

We now pursue the degree theoretic approach to the question of the refinement of the GNE in the primal-dual space. Throughout, we assume an algebraic form for the constraint set \mathbb{C} and assume that an appropriate constraint qualification holds. The nonlinear equations whose zero is the GNE (in the primal-dual space), are, expectedly, given by the natural map of a complementarity problem (CP). In our formulation, the nonlinear equations for the VE are the natural map of a CP with

2.3 Primal-dual generalized Nash and variational equilibria

additional linear constraints (the reason for this will be clarified below). Using these maps, we show that the degree theoretic approach of the previous section has a natural extension to the primal-dual space. While this extension may be intuitively expected, it must be emphasized that it is not a corollary of the primal approach. The maps used in the primal setting ($\mathbf{F}_C^{\text{nat}}$ and $\tilde{\mathbf{F}}_K^{\text{nat}}$) and the ones we use in the primal-dual setting (\mathbf{G}^{nat} and \mathbf{J}^{nat}) are very different (and defined on different spaces) and there is no obvious degree theoretic connection that one can draw between them.

The primal-dual characterization offers various advantages over the primal characterization. For instance, the assumptions are less abstract and easier to verify; the constraints are assumed to have continuously differentiable boundaries of algebraic form, as opposed to the assumption of continuity of the set-valued map K . The analysis is simplified since we escape the QVI setting and now work with pure CPs or mixed CPs. As a result the well-formed theory of VIs and CPs can be used to our advantage. Moreover the set over which these CPs are defined are cartesian products of sets (in the primal-dual space), i.e. these CPs are *cartesian* VIs or *partitioned* VIs for which numerous results are known that are simpler than those for general VIs.

We assume, unless mentioned otherwise, that $\mathbb{C} = \{x \mid x \geq 0, c(x) \geq 0\}$ where $c : \mathbb{R}^m \rightarrow \mathbb{R}$ is a concave continuously differentiable function. c is assumed \mathbb{R} -valued as opposed to \mathbb{R}^n -valued, only to ease the exposition; in most cases, no generality is lost. We will make the generality or the absence thereof clear wherever necessary. Recall optimization problems A_i from Section 2.1. Suppose an appropriate constraint qualification holds (see [FP03]). Then the Karush-Kuhn-Tucker (KKT) conditions for optimality of x_i for $A_i(x^{-i})$ are given by

$$\begin{aligned} 0 &\leq x_i \perp \nabla_i \varphi_i(x) - \lambda_i \nabla_i c(x) \geq 0 \\ 0 &\leq \lambda_i \perp c(x) \geq 0, \end{aligned} \tag{KKT}_i$$

where λ_i are the Lagrange multipliers for A_i corresponding to the constraint $c(\cdot) \geq 0$. For $u, v \in \mathbb{R}^n$, the notation $0 \leq u \perp v \geq 0$ means $u, v \geq 0$ and $u_i v_i = 0$ for $i = 1, \dots, n$. A strategy tuple x is a GNE of \mathcal{G} if there exist $\lambda_1, \dots, \lambda_N$ that together with x simultaneously satisfy the N systems $\{\text{KKT}_i\}_{i \in \mathcal{N}}$. The VE can also be interpreted in terms of the KKT systems, as observed by Facchinei et al. [FFP07].

Theorem 2.12 (Theorem 3.1 [FFP07]) *Let x be a GNE for which the system $\{\text{KKT}_i\}_{i \in \mathcal{N}}$ is*

Chapter 2. Refinement of the Generalized Nash Equilibrium

satisfied with $\lambda_1 = \lambda_2 = \dots = \lambda_N$. Then x solves $\text{VI}(\mathbb{C}, F)$. Conversely if x solves $\text{VI}(\mathbb{C}, F)$ then there exist $\lambda \in \mathbb{R}$ such that $\{\text{KKT}_i\}_{i \in \mathcal{N}}$ hold with $\lambda_i = \lambda$ for all $i \in \mathcal{N}$.

Proof : The proof follows exactly the same argument as in Theorem 3.1 [FFP07]. ■

This theorem sheds new light on equilibria with “shared multipliers” and “non-shared multipliers” for games with shared constraints. Historically, if there existed $(x, \lambda_1, \dots, \lambda_N) \in \mathbb{R}^{m+N}$ that satisfied $\{\text{KKT}_i\}_{i \in \mathcal{N}}$, x was called a GNE of \mathcal{G} with *non-shared multipliers*. If for some $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$, $\{\text{KKT}_i\}_{i \in \mathcal{N}}$ were met with $\lambda_1 = \lambda_2 = \dots = \lambda_N = \lambda$, then x was called a GNE of \mathcal{G} with *shared multipliers*. Rosen [Ros65] was the first to spot the possibility of redundancy of multipliers when the constraint was shared – he termed the shared multiplier GNE as a *normalized equilibrium*. The fact that the GNE with shared multipliers was actually a solution of a related VI was a new insight in the context of such games.

With this understanding, the idea of the VE as a refinement of GNE can now be cast differently. The class of games \mathcal{S} for which the implication in (2.3) holds are those for which the existence of GNE with non-shared multipliers implies the existence of a GNE with shared multipliers. This property also has an interesting economic interpretation. In many games with shared constraints, such as a bandwidth sharing game [JT04], the Lagrange multipliers corresponding to the constraint can be interpreted as the “price” charged on a player for using a particular strategy by an administrator who controls the shared constraint. Thus the equilibrium with non-shared multipliers can be interpreted as an equilibrium resulting from discriminatory prices, i.e. prices that are charged by an administrator who may discriminate between various users of his constraint. But often the situation modeled makes it unrealistic for the administrator to be able to distinguish between users and the economically appropriate equilibrium is one in which the *same* price is charged to all users, i.e. a shared multiplier equilibrium. The question of the refinement assumes immediate relevance here. When does an administrator always have the option of charging a uniform price across all users to enforce equilibrium? If the implication in (2.3) holds, we can say that whenever an equilibrium with discriminatory prices exists, one with uniform prices also exists. Moreover, if an equilibrium with uniform prices does not exist, *no* equilibrium exists.

2.3 Primal-dual generalized Nash and variational equilibria

■ 2.3.1 Refinement of the primal-dual GNE

A result similar to Theorem 2.5 is obtainable by invoking KKT conditions for characterizing GNE and VE. Recall the systems $(\text{KKT})_i$, denote by Λ the tuple $(\lambda_1, \dots, \lambda_N)$ and define \mathbf{G}^{nat} and $\mathbf{J}^{\text{nat}} : \mathbb{R}^{m+N} \rightarrow \mathbb{R}^{m+N}$ as follows.

$$\mathbf{G}^{\text{nat}}(x, \Lambda) := \begin{pmatrix} x_1 - \Pi_+(x_1 - \nabla_1 \varphi_1(x) + \lambda_1 \nabla_1 c(x)) \\ \vdots \\ x_N - \Pi_+(x_N - \nabla_N \varphi_N(x) + \lambda_N \nabla_N c(x)) \\ \lambda_1 - \Pi_+(\lambda_1 - c(x)) \\ \vdots \\ \lambda_N - \Pi_+(\lambda_N - c(x)) \end{pmatrix},$$

$$\mathbf{J}^{\text{nat}}(x, \Lambda) := \begin{pmatrix} x_1 - \Pi_+(x_1 - \nabla_1 \varphi_1(x) + \lambda_1 \nabla_1 c(x)) \\ \vdots \\ x_N - \Pi_+(x_N - \nabla_N \varphi_N(x) + \lambda_N \nabla_N c(x)) \\ \lambda_1 - \Pi_+(\lambda_1 - c(x)) \\ \lambda_2 - \lambda_1 \\ \vdots \\ \lambda_N - \lambda_1 \end{pmatrix},$$

$\forall x \in \mathbb{R}^m, \Lambda \in \mathbb{R}^N$, where $\Pi_+(\cdot)$ is the Euclidean projection on the nonnegative orthant of appropriate dimension. Recall that the relation $0 \leq u \perp v \geq 0$ is equivalent to $u = \Pi_+(u - v)$ [FP03]. It follows from comparison with KKT_i 's that x solves $\text{QVI}(K, F)$ if and only if there exists $\Lambda \in \mathbb{R}^N$ such that $\mathbf{G}^{\text{nat}}(x, \Lambda) = 0$. By Theorem 2.12, x solves $\text{VI}(\mathbb{C}, F)$ if and only if there exists $\Lambda \in \mathbb{R}^N$ such that $\mathbf{J}^{\text{nat}}(x, \Lambda) = 0$. Notice the structure of \mathbf{J}^{nat} : contrary to popular approaches to characterizing the *shared* multiplier that express the shared multiplier as a point in \mathbb{R} , we treat the shared multiplier explicitly as a vector in \mathbb{R}^N with identical coordinates. For the result below, it is analytically convenient to have equation reformulations of the GNE and VE with the same domain (\mathbb{R}^{m+N}) .

Chapter 2. Refinement of the Generalized Nash Equilibrium

Theorem 2.13 *Let Ω be an open bounded set in \mathbb{R}^{m+N} such that $0 \notin \mathbf{G}^{\text{nat}}(\partial\Omega)$. Then*

$$\deg(\mathbf{G}^{\text{nat}}, \Omega, 0) = \deg(\mathbf{J}^{\text{nat}}, \Omega, 0).$$

Proof : Observe that since $0 \notin \mathbf{G}^{\text{nat}}(\partial\Omega)$, \mathbf{J}^{nat} is not zero on $\partial\Omega$ and $\deg(\mathbf{G}^{\text{nat}}, \Omega, 0)$ and $\deg(\mathbf{J}^{\text{nat}}, \Omega, 0)$ are well defined. We again will invoke the homotopy invariance of the Brouwer degree. Define $H : [0, 1] \times \bar{\Omega} \rightarrow \mathbb{R}^{m+N}$ as

$$H(\bar{t}, \bar{x}, \bar{\Lambda}) = \bar{t}\mathbf{G}^{\text{nat}}(\bar{x}, \bar{\Lambda}) + (1 - \bar{t})\mathbf{J}^{\text{nat}}(\bar{x}, \bar{\Lambda}) \quad \forall \bar{t} \in [0, 1], (\bar{x}, \bar{\Lambda}) \in \bar{\Omega}.$$

We will show that $\deg(H(\bar{t}, \cdot), \Omega, 0)$ is well defined for each $\bar{t} \in [0, 1]$ and then invoke its invariance with respect to \bar{t} to conclude the claim. We know that this degree is well defined for $t \in \{0, 1\}$. Assume that it is not so for some $t \in (0, 1)$, i.e. assume that for some $t \in (0, 1)$ and $(x, \Lambda) \in \partial\Omega$, $H(t, x, \Lambda) = 0$. Therefore

$$\begin{aligned} x_i - \Pi_+(x_i - \nabla_i \varphi_i(x) + \nabla_i c(x)\lambda_i) &= 0 \quad \forall i \in \mathcal{N}, \\ \lambda_1 - \Pi_+(\lambda_1 - c(x)) &= 0 \end{aligned} \tag{2.14}$$

$$t\Pi_+(\lambda_i - c(x)) + (1 - t)\lambda_1 = \lambda_i. \quad \forall i \in \mathcal{N} \setminus \{1\}. \tag{2.15}$$

It follows that $(x, \Lambda) \in \mathbb{R}_+^{m+N}$. From (2.14) it is clear that $0 \leq \lambda_1 \perp c(x) \geq 0$. Pick an arbitrary $i \neq 1$. We will show that $0 \leq \lambda_i \perp c(x) \geq 0$ holds for this i . From (2.15) it follows that $\lambda_i \geq 0$. Since λ_i is a real number, two cases arise:

(a) If $\lambda_i \geq c(x)$, then $\Pi_+(\lambda_i - c(x)) = \lambda_i - c(x)$. So we get from (2.15)

$$(1 - t)(\lambda_i - \lambda_1) = -tc(x) \implies (1 - t)\lambda_i c(x) = -tc(x)^2 \implies \lambda_i c(x) = 0,$$

where the last implication is deduced from noting that $c(x) \geq 0$, $\lambda_i \geq 0$ and $t > 0$.

(b) On the other hand if $\lambda_i < c(x)$,

$$\lambda_i = (1 - t)\lambda_1 \implies \lambda_i c(x) = 0.$$

2.3 Primal-dual generalized Nash and variational equilibria

Hence in either case, $0 \leq \lambda_i \perp c(x) \geq 0$ or $\lambda_i = \Pi_+(\lambda_i - c(x))$. and so from (2.15), $\lambda_i = \lambda_1$. As i was arbitrary, we have $\lambda_1 = \lambda_2 = \dots = \lambda_N$. But this means $\mathbf{J}^{\text{nat}}(x, \Lambda) = 0$, a contradiction to our hypothesis.

So $0 \notin H(\bar{t}, \partial\Omega)$ for all \bar{t} in $[0, 1]$ and $\deg(H(\bar{t}, \cdot), \Omega, 0)$ is independent of \bar{t} . The claim follows. ■

Remark : The above theorem can be generalized for games where $c : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $n > 1$. This would merely require steps (a) and (b) above to be repeated for each of the n components of $c(x)$. □

Theorem 2.13 plays the same role in the primal-dual space as Theorem 2.5 did in the primal space, insofar as studying the VE as a refinement of the GNE. By an argument analogous to that in Theorem 2.6, the VE is a refinement for the class of games for which the existence of a GNE implies that we can find an Ω as in Theorem 2.13 so that $\deg(\mathbf{G}^{\text{nat}}, \Omega, 0) \neq 0$. In the remainder of this section we will develop a condition that ensures this.

Let $\psi : \mathbb{R}^{m+N} \rightarrow \mathbb{R}^{m+N}$ be defined as follows

$$\psi(x, \Lambda) := \begin{pmatrix} \nabla_1 \varphi_1(x) - \lambda_1 \nabla_1 c(x) \\ \vdots \\ \nabla_N \varphi_N(x) - \lambda_N \nabla_N c(x) \\ c(x) \\ \vdots \\ c(x) \end{pmatrix} \quad (N \text{ times})$$

Observe that \mathbf{G}^{nat} is the natural map of $\text{VI}(\mathbb{R}_+^{m+N}, \psi)$: $\mathbf{G}^{\text{nat}}(z) = z - \Pi_+(z - \psi(z))$. Moreover, notice that $\text{VI}(\mathbb{R}_+^{m+N}, \psi)$ is a *cartesian* VI in the sense of [FP03] over the following set

$$\mathbb{R}_+^{m+N} = \prod_{k=1}^{2N} \mathbb{R}_+^{\nu_k}, \quad \text{where } \nu_k = \begin{cases} m_k & k = 1, \dots, N \\ 1 & k = N+1, \dots, 2N. \end{cases} \quad (2.16)$$

Suppose ψ is a \mathbf{P}_0 function and $\text{VI}(\mathbb{R}_+^{m+N}, \psi)$ has a bounded solution set. It can be shown that $\deg(\mathbf{G}^{\text{nat}}, \Omega, 0) = \pm 1$ for any open bounded set Ω that contains $\text{SOL}(\text{VI}(\mathbb{R}_+^{m+N}, \psi))$ [FP03, section 3.6.1]. Combining with Theorem 2.13, we see that such a game also admits a VE. This is

Chapter 2. Refinement of the Generalized Nash Equilibrium

articulated in the following theorem. The definition of a \mathbf{P}_0 mapping is as follows. $\psi : \mathbb{R}_+^{m+N} \rightarrow \mathbb{R}^{m+N}$ is said to \mathbf{P}_0 on \mathbb{R}_+^{m+N} partitioned as in (2.16) if for all $x, y \in \mathbb{R}^{m+N}$, there exists an index $i \in \{1, \dots, 2N\}$ such that $x_i \neq y_i$ and $(\psi_i(x) - \psi_i(y))^T(x_i - y_i) \geq 0$.

Theorem 2.14 *Suppose ψ is a \mathbf{P}_0 mapping and $(\mathbf{G}^{\text{nat}})^{-1}(0)$ (if nonempty) is bounded, then the implication*

$$(\mathbf{G}^{\text{nat}})^{-1}(0) \neq \emptyset \implies (\mathbf{J}^{\text{nat}})^{-1}(0) \neq \emptyset,$$

holds and this game admits the VE as a refinement of the GNE.

Note that ψ being \mathbf{P}_0 is not sufficient for $\text{VI}(\mathbb{R}_+^{m+N}, \psi)$ to have a solution and \mathcal{G} to have a GNE. But if a GNE exists, and the set of GNEs is bounded (in the primal-dual space), then the above theorem shows that a VE also exists. Consequently, the game in the above result belongs to $\mathcal{S} \setminus \mathcal{S}'_2$. A sufficient condition for the boundedness of $\text{SOL}(\text{VI}(\mathbb{R}_+^{m+N}, \psi))$ can be seen Theorem 5.5.15 in [FP03].

We conclude this section by giving a sufficient condition for ψ to be a \mathbf{P}_0 function. Recall that ψ is \mathbf{P}_0 if its Jacobian, $\nabla\psi(z)$, is a \mathbf{P}_0 matrix for any z . Lemma 2.15 gives a sufficient condition for $\nabla\psi$ to be a \mathbf{P}_0 matrix. Assume that c is twice continuously differentiable and let the Jacobian of ψ be defined as

$$\Psi(x, \Lambda) := \nabla\psi = \begin{pmatrix} \mathbf{H} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{pmatrix},$$

where submatrices $\mathbf{H}, \mathbf{B}, \mathbf{C}, \mathbf{0}$ are as indicated below.

$$\left[\begin{array}{ccc|ccc} \nabla_{11}\varphi_1(x) - \nabla_{11}c(x)\lambda_1 & \cdots & \nabla_{1N}\varphi_1(x) - \nabla_{1N}c(x)\lambda_1 & -\nabla_1c(x) & \cdots & 0 \\ & \ddots & & 0 & \ddots & [\mathbf{B}] & 0 \\ \nabla_{N1}\varphi_N(x) - \nabla_{1N}c(x)\lambda_N & \cdots & \nabla_{NN}\varphi_N(x) - \nabla_{NN}c(x)\lambda_N & 0 & \cdots & -\nabla_Nc(x) \\ \hline \nabla_1c(x)^T & \cdots & \nabla_Nc(x)^T & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & \ddots & [\mathbf{0}] & 0 \\ \nabla_1c(x)^T & \cdots & \nabla_Nc(x)^T & 0 & \cdots & 0 \end{array} \right].$$

Lemma 2.15 *Let $c : \mathbb{R}^m \rightarrow \mathbb{R}$ be a concave function in C^2 . Assume that for all $x, \Lambda \geq 0$ $\mathbf{H}(x, \Lambda)$ is a block diagonal positive definite matrix with blocks $\mathbf{H}_{1,1}, \dots, \mathbf{H}_{N,N}$ where for each $i \in \mathcal{N}$, the*

2.3 Primal-dual generalized Nash and variational equilibria

submatrix $\mathbf{H}_{i,i}$ is a positive definite matrix in $\mathbb{R}^{m_i \times m_i}$. Then $\Psi(x, \Lambda)$ is a \mathbf{P}_0 matrix.

Proof : The proof follows by showing that every principal submatrix of $\Psi(x, \Lambda)$ has a nonnegative determinant. Consider an arbitrary submatrix D is given by $D = \Psi(x, \Lambda)_{\alpha, \alpha}$ where $\alpha \subseteq \{1, \dots, m+N\}$ is an index set. Let $\alpha = \beta \cup \gamma$, where $\beta \subseteq \{1, \dots, m\}, \gamma \subseteq \{m+1, \dots, m+N\}$. Then $\Psi(x, \Lambda)_{\alpha, \alpha}$ is given by

$$\begin{pmatrix} \mathbf{H}(x, \Lambda)_{\beta, \beta} & \mathbf{B}(x)_{\beta, \gamma} \\ \mathbf{C}(x)_{\gamma, \beta} & \mathbf{0}_{\gamma, \gamma} \end{pmatrix},$$

We drop arguments (x) and (x, Λ) for brevity. Consider some $\beta \subseteq \{1, \dots, m\}$. If $[\mathbf{C}_{\gamma, \beta}, \mathbf{0}_{\gamma, \gamma}]$ has at least 2 identical rows it follows that $\det(D) = 0$. Since c is \mathbb{R} -valued, it suffices to consider the case where $|\gamma| \leq 1$ and $[\mathbf{C}_{\gamma, \beta}, \mathbf{0}_{\gamma, \gamma}]$ does not contain all zeros. For any $i \in \beta$ let $b_i := \min\{k \mid i \leq \sum_{j=1}^k m_j\}$ and let $\kappa_i = m + b_i$. For a row $i \in \beta$, κ_i is the column of \mathbf{B} that contains the vector $-\nabla_j c$ through which row i passes. i.e. if $\mathbf{B}[i, j]$ denotes the element in the i^{th} row and $j - m^{\text{th}}$ column of \mathbf{B} , we have,

$$j \neq \kappa_i \implies \mathbf{B}[i, j] = 0.$$

Recall that γ has at most one element. Based on γ three cases arise:

1. $\gamma = \emptyset$: In this case D is a principal submatrix of \mathbf{H} . Since $\mathbf{H} \succ 0$, it follows that $\det(D) \geq 0$.
2. $\gamma \neq \emptyset, \gamma \cap \bigcup_{i \in \beta} \kappa_i = \emptyset$: Let $\gamma = \{j\}$. This assumption ensures that for all $i \in \beta$, $j \neq \kappa_i$. Hence there is a column of $\mathbf{B}_{\beta, \gamma}$ that has all zeros. Consequently there is a zero column of D . Hence $\det(D) = 0$.
3. $\gamma \neq \emptyset, \gamma \cap \bigcup_{i \in \beta} \kappa_i = \gamma$: Let $\gamma = \{j\}$ This means that there is an $i \in \beta$ such that $j = \kappa_i$. Assume that $j = m + 1$ and $\beta = \{1, \dots, m\}$. We shall see that there is no loss of generality in this assumption. Then, recalling that \mathbf{H} is block diagonal, D may be written as

$$D = \begin{pmatrix} \mathbf{H}_{1,1} & \dots & \mathbf{0}_{1,N} & -\nabla_1 c \\ & \ddots & & 0 \\ \mathbf{0}_{N,1} & \dots & \mathbf{H}_{N,N} & 0 \\ \nabla_1 c^T & \dots & \nabla_N c^T & 0 \end{pmatrix}.$$

Chapter 2. Refinement of the Generalized Nash Equilibrium

where $\mathbf{H}_{i,i} \in \mathbb{R}^{m_i \times m_i}$. Using the Schur complement [HJ90] we may write the determinant of D as

$$\begin{aligned} \det(D) &= \det(\mathbf{H}) \det(-\mathbf{C}_{\gamma,\beta} \mathbf{H}^{-1} \mathbf{B}_{\beta,\gamma}) \\ &= \det(\mathbf{H}) \det(\nabla_1 c^T \mathbf{H}_{1,1}^{-1} \nabla_1 c) \geq 0. \end{aligned}$$

Since c is \mathbb{R} -valued, $\nabla_1 c$ is a vector. The nonnegativity of $\det(D)$ follows from the positive definiteness of \mathbf{H} and of the inverse of $\mathbf{H}_{1,1}$.

It is easy to see that the above arguments would hold if we picked γ to comprise some other element ($\neq m+1$) and $\mathbf{H}_{\beta,\beta}$ was any other principal submatrix of \mathbf{H} . ■

Remark : The above result has assumed that c is \mathbb{R} -valued. Extending this result to higher dimensions may require making stronger assumptions on the properties of the Jacobian, so as to ensure that the resulting Schur complement is a \mathbf{P}_0 matrix. □

An example of a game where the hypotheses of Lemma 2.15 are satisfied is network routing game with affine coupling constraints considered in [JT04]. We present a modification of this game below.

Example 2.5. A network routing game: Assume each player has real valued strategies and solves an optimization problem given by

$\mathbf{A}_i(x^{-i})$	$\begin{aligned} &\underset{x_i}{\text{minimize}} && \varphi_i(x_i; x^{-i}) = \mathcal{U}_i(x_i) \\ &\text{subject to} && a^T x \geq b \quad : \lambda_i, \\ & && x \geq 0, \end{aligned}$
------------------------	--

where $a \in \mathbb{R}^m, b \in \mathbb{R}$ and \mathcal{U}_i is a convex continuously differentiable function. This is clearly a shared constraint game with $m_i = 1$ for all $i \in \mathcal{N}$ and $\mathbb{C} = \{x \mid a^T x \geq b\}$. If \mathcal{U}_i are strictly convex for each $i \in \mathcal{N}$, F is a strictly monotone function and \mathbf{H} is a positive definite diagonal matrix. Ordinarily, this would not be sufficient to claim that $\text{VI}(\mathbb{C}, F)$ has a solution and that this game has a VE. But if given that this game has a bounded and nonempty set of GNEs (in the primal-dual

2.4 Refinement of the GNE in power markets

space), one may use Theorem 2.13 and Lemma 2.15 to conclude that a VE exists for this game. In other words, for this game, the VE is a refinement of the GNE and this game belongs to the class $\mathcal{S} \setminus \mathcal{S}'_2$. In fact, using the strict monotonicity of F , we may also claim that the VE is unique. Furthermore, if we are independently given the boundedness of the (possibly empty) set of GNEs in the primal-dual space, we may say that the game either admits a VE or admits no GNE at all. This is also compatible with economic intuition that for such games, the equilibrium with uniform prices is the more appealing solution concept and is the one applied in [JT04]. This is another example where our results substantiate an idea that was previously had an intuitive appeal, but no formal justification. \square

■ 2.4 Refinement of the GNE in power markets

In this section, we show that a shared-constraint game arising in power markets, admits the VE as a refinement of its GNE. The model is fairly broad and may also be applied to a game of capacity expansion [MS03] with demand constraints. The game is distinguished by its model which depicts Cournot competition. Since the constraint of serving demand is “shared” by all players, this game is a shared-constraint game.

Consider a power market comprising of firms $\mathcal{N} = \{1, \dots, N\}$ competing over an electricity network comprising of a set of nodes, denoted by \mathbf{N} . The j^{th} firm may own generation facilities at a subset of nodes denoted by \mathbf{N}_j and we denote the set of firms with such facilities at node i by \mathcal{N}_i . The objective of generating firms is to choose their generation level so as to maximize their profit subject to the sum of their generations being sufficient to meet the demand.

In the market we assume, the price of electricity at the nodes is determined via a Cournot model, i.e. the price is blind to the generation levels of individual generators present at the node and is dependent only on the sum of their generations. Specifically, if $\hat{x}_i := (x_{ji})_{j \in \mathcal{N}_i}$ are the generation levels at node i , p_i , the price at node i , is given by

$$p_i(\hat{x}_i) := a_i - b_i \sum_{j \in \mathcal{N}_i} x_{ji}, \quad \forall i \in \mathbf{N}. \quad (2.17)$$

Finally, let the cost of generation of quantity x_{ji} for firm j at node i be $\zeta_{ji}(x_{ji})$, whereby the loss,

Chapter 2. Refinement of the Generalized Nash Equilibrium

φ_j of firm j is given by :

$$\varphi_j(x_j; x^{-j}) := - \sum_{i \in \mathbf{N}_j} (p_i(\hat{x}_i) x_{ji} - \zeta_{ji}(x_{ji})), \quad (2.18)$$

where x_j is defined as $x_j := (x_{ji})_{i \in \mathbf{N}_j}$. Unlike before, we follow a special convention in denoting the tuple of strategies of all players. Let $n = |\mathbf{N}|$ be the number of nodes in the network. The tuple of strategies x is denoted by $(\hat{x}_1, \dots, \hat{x}_n)$ and for a firm $j \in \mathcal{N}$, we denote the tuple $x^{-j} := (\hat{x}_1^{-j}, \dots, \hat{x}_n^{-j})$ where $\hat{x}_i^{-j} := (x_{ki})_{k \in \mathcal{N}_i \setminus j}$. If d_i denotes the demand at node i , the *shared constraint* that the firms' generations are required to satisfy is

$$\mathbb{C} := \left\{ x \mid x \geq 0, \sum_{j \in \mathcal{N}_i} x_{ji} \geq d_i, \forall i \in \mathbf{N} \right\}. \quad (2.19)$$

As before we denote the feasible region of generation firm j 's optimization problem as

$$K_j(z^{-j}) := \{z \mid (z, z^{-j}) \in \mathbb{C}\}, \quad \forall j \in \mathcal{N},$$

and let $K = \prod_{j \in \mathcal{N}} K_j$. The resulting maximization problem faced by the j^{th} firm is given by

$\begin{aligned} & \text{minimize}_{x_j} \quad \varphi_j(x_j; x^{-j}) \\ & \text{subject to} \quad x_j \in K_j(x^{-j}). \end{aligned}$
--

The game resulting out of problems A_j is clearly a shared-constraint game. The GNE of this game is often referred to as the *Nash-Cournot* equilibrium, with an additional demand constraint. The variational equilibrium (VE) of this game is given by the solution to $\text{VI}(\mathbb{C}, F)$ where $F := ((F_1)^T, \dots, (F_n)^T)^T$, where

$$F_i(\hat{x}_i) = \left[\frac{\partial \varphi_j(x_j; x^{-j})}{\partial x_{ji}} \right]_{j \in \mathcal{N}_i}.$$

2.4 Refinement of the GNE in power markets

Proposition 2.16 *Let $x^{\text{ref}} \in \partial\mathbb{C}$ be a GNE of the above game with*

$$F(x^{\text{ref}}) \in -\text{int}(\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}}))).$$

If the generation cost ζ_{ji} are convex for all $j \in \mathcal{N}, i \in \mathbf{N}$ and then the game \mathcal{G} admits a VE.

Proof : Let $m_i = |\mathcal{N}_i|$ be the number of firms with generation at node i and $m = \sum_{i \in \mathbf{N}} m_i$. Thus the tuple of the strategies of all firms is a vector in \mathbb{R}^m . We begin by observing that $\mathbb{C} = \{x \in \mathbb{R}^m \mid x \geq 0, Ax \geq d\}$, where A is a $n \times m$ matrix, d is a n -dimensional vector of demands, $[d_1, \dots, d_n]$. Every element of A is either 0 or 1, according to the following rule:

$$A[i, j] = \begin{cases} 1 & \text{if } \sum_{k=0}^{i-1} m_k < j \leq \sum_{k=0}^i m_k \\ 0 & \text{else,} \end{cases}$$

where we define $m_0 := 0$. By the same argument as in Example 3, we get $\mathbb{C}_\infty = \mathcal{T}(x^{\text{ref}}; K(x^{\text{ref}}))$, whereby condition (2) from Theorem 2.11 holds. And since $F(x^{\text{ref}}) \in -\text{int}(\mathcal{N}(x^{\text{ref}}; K(x^{\text{ref}})))$, condition (1) from Theorem 2.11 also holds. To show the result, it remains to show that F is pseudo-monotone, which holds if F is monotone. From above,

$$F_i(\hat{x}_i) = - (p_i(\hat{x}_i) - b_i x_{ji} - \zeta'_{ji}(x_{ji}))_{j \in \mathcal{N}_i},$$

and therefore

$$\nabla F_i(\hat{x}_i) := \text{diag} [b_i + \zeta''_{ji}(x_{ji})]_{j \in \mathcal{N}_i} + b_i E_i,$$

where $E_i = e_i e_i^T$ and e_i an m_i -dimensional vector with each element 1. Since φ_j 's are nodally separable, we get $\nabla F = \text{diag}[\nabla F_i]_{i \in \mathcal{N}}$. It is now easy to see that ∇F is positive semidefinite. Observe that

$$s^T E_i s = s^T e_i e_i^T s = (\sum s_j)^2 \geq 0,$$

and consequently E_i is a positive semidefinite matrix. Since ζ_{ji} 's are convex and b_i is nonnegative, ∇F_i is a sum of two positive semidefinite matrices, from which the result follows. ■

■ 2.5 Conclusions

In this chapter, we presented a theory of the VE as a refinement of the GNE. The GNE and the VE were shown to be related via a Brouwer degree-theoretic equivalence in the primal and primal-dual space. These equivalences led to sufficiency conditions for a shared-constraint game to have the VE as a refinement of the GNE. Furthermore, for certain games these conditions were seen as necessary. Finally, this framework was applied on a class of Nash-Cournot games in power market games where it was shown that the VE was indeed a refinement of the GNE.

■ 2.6 Supplementary examples and results

This section contains some examples, results and proofs that serve as to supplement the arguments made in this chapter. We begin with some examples of shared-constraint games and then give a sufficient condition for a game to have a manifold of GNEs. Finally we include proofs of Lemmas 2.2 and 2.3.

■ 2.6.1 Some examples of shared-constraint games

In this section, we construct examples that illustrate the unusual properties shared-constraint games can exhibit. Often these properties are contributed by the structure of the set-valued map K , particularly on points on the boundary of the domain of K . Recall Theorem 2.5 where we showed that $\deg(\tilde{\mathbf{F}}_K^{\text{nat}}, \Omega, 0) = \deg(\mathbf{F}_{\mathbb{C}}^{\text{nat}}, \Omega, 0)$, where Ω was an open bounded set inside $\text{dom}(K)$. Theorem 2.5 applies only when Ω is taken from the interior of $\text{dom}(K)$. We will see in the examples below that on the boundary of $\text{dom}(K)$, $\text{QVI}(K, F)$ can show surprising behaviour. Throughout we assume $N = 2$ and $m_1 = m_2 = 1$.

Example 2.6. Game with GNE independent of F : Often the nature of K (and \mathbb{C}) plays a dominant role in determining $\text{SOL}(\text{QVI}(K, F))$, in the sense that it renders some points as solutions regardless of F . Fig. 2.3 shows \mathbb{C} and a point $x^* \in \partial\mathbb{C}$ with the property that for *any* F , x^* solves $\text{QVI}(K, F)$. This is because the image of x^* under K is a singleton, namely x^* itself. In Fig. 2.3, dotted lines depict axes with their origin shifted to x^* . If $y \in K(x^*)$, the points (y_1, x_2^*) and (x_1^*, y_2) lie in on these ‘axes’. Notice that since these ‘axes’ intersect \mathbb{C} at only one point, x^* , $K(x^*) = \{x^*\}$.

2.6 Supplementary examples and results

As a result for any F , we have

$$F(x^*)^T(y - x^*) = 0 \quad \forall y \in K(x^*)$$

implying that $x^* \in \text{SOL}(\text{QVI}(K, F))$. Observe that x^* lies in $\partial \text{dom}(K)$. □

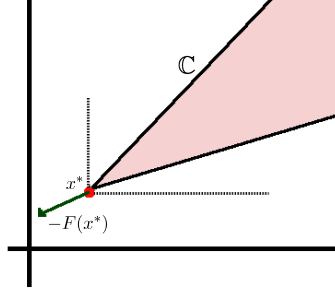


Figure 2.3: An example where $K(x^*) = \{x^*\}$. Independently of F , x^* solves $\text{QVI}(K, F)$. When x^* is the origin and $F(x) = (x_1, 1)$, x^* is the only solution.

Example 2.7. Game with unique GNE and VE: While sufficiency conditions for a manifold of solutions to a QVI were provided in Theorem 2.17, QVIs with unique solutions also exist. Suppose x^* in Fig. 2.3 is the origin of the coordinate system, i.e. let $x^* = (0, 0)$. Assume that $\mathbb{C} = \{(x_1, x_2) \mid x \geq 0, x_1 \in [x_2, 2x_2]\}$, so that

$$K(x) = \{(y_1, y_2) \mid y \geq 0, y_1 \in [x_2, 2x_2], y_2 \leq x_1 \leq 2y_2, x \geq 0\}.$$

Let $F(x) = (x_1, 1)$ (arising from, say, $\varphi_1(x) = \frac{1}{2}x_1^2 + x_2, \varphi_2(x) = x_2 - x_1$). This game has a unique GNE, x^* . Indeed one may verify that $K(x^*) = \{x^*\}$, and conclude from the previous example that x^* is a solution for any F . To see that this is the only solution, suppose $x \in \mathbb{C} \setminus \{x^*\}$ solves $\text{QVI}(K, F)$. i.e.

$$\begin{pmatrix} x_1 \\ 1 \end{pmatrix}^T \left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \geq 0 \quad \forall y \in K(x).$$

Observe that $y = (x_1, \frac{1}{2}x_1) \in K(x)$ since $x_1 \in [x_2, 2x_2]$. Substituting above gives $x_1 \geq 2x_2$, which implies the solution must satisfy $x_1 = 2x_2$. Now observe that $y = (x_2, \frac{1}{2}x_1) = (\frac{1}{2}x_1, \frac{1}{2}x_1) \in K(x)$. Substituting this y above gives $\frac{1}{2}x_1^2 \leq 0$. It follows that $x = (0, 0) = x^*$, a contradiction. That

makes x^* the only solution. Notice that Theorem 2.17 does not apply here because $(x^*, \Lambda^*) \not\geq 0$. It is easy to check that x^* also solves $\text{VI}(\mathbb{C}, F)$, i.e. x^* is a VE, and is the unique VE. Thus, for this game, $\text{SOL}(\text{VI}(\mathbb{C}, F)) = \text{SOL}(\text{QVI}(K, F))$. \square

■ 2.6.2 On the existence of a manifold of GNEs

A remark made in [FFP07] claimed that generalized Nash games often have a manifold of equilibria. In Example 2, we saw a QVI with a unique solution. Theorem 2.17 below shows that manifolds may exist under some conditions on the dimensions of the QVI. Here we assume $c : \mathbb{R}^m \rightarrow \mathbb{R}^n$ to be a C^1 concave function and that an appropriate constraint qualification holds.

Theorem 2.17 *Consider a game in which $m = nN$. Let $(x^*, \Lambda^*) > 0$ be a GNE of such a game such that the square matrix $\mathbf{B}(x^*)$ is nonsingular. Then there exists a neighbourhood $B(x^*, r) \subseteq \mathbb{R}^m$ of x^* of radius r such that for every $x \in B(x^*, r) \cap \{v \mid c(v) = 0\}$, there exists $\Lambda \geq 0$ so that (x, Λ) is a GNE.*

Proof : Since $(x^*, \Lambda^*) > 0$ it is easy to see from the KKT conditions and nonsingularity of $\mathbf{B}(x^*)$ that

$$\Lambda^* = -\mathbf{B}(x^*)^{-1}F(x^*).$$

$\det(\mathbf{B}(\cdot))$ is a $\mathbb{R}^m \rightarrow \mathbb{R}$ continuous function. By continuity there is a neighbourhood $B(x^*, r_1) \subseteq \mathbb{R}^m$ of x^* such that $\text{sgn } \det(\mathbf{B}(x)) = \text{sgn } \det(\mathbf{B}(x^*))$ for all $x \in B(x^*, r_1)$. Thus $\mathbf{B}(x)$ is nonsingular on $B(x^*, r_1)$. Furthermore since $-\mathbf{B}(x^*)^{-1}F(x^*) > 0$ and $x^* > 0$ there is another neighbourhood $B(x^*, r_2)$ of x^* such that for all $x \in B(x^*, r_2)$, $-\mathbf{B}(x)^{-1}F(x) \geq 0$ and $x > 0$.

Finally, let $r = \min\{r_1, r_2\}$ and pick an arbitrary $x \in B(x^*, r) \cap \{v \mid c(v) = 0\}$. Since $r \leq r_2$, $x > 0$. Using this it is easy to see that for this x , the pair (x, Λ) , where $\Lambda = -\mathbf{B}(x)^{-1}F(x)$, is a GNE. \blacksquare

Remark : Theorem 2.17 can be extended to the case where $m < nN$ by replacing the hypothesis of nonsingularity of $\mathbf{B}(x^*)$ with one of full row-rank. \square

2.6 Supplementary examples and results

■ 2.6.3 Proof Lemma 2.2

Proof :

1. Take any $i \in \mathcal{N}$ and consider an $x \in \mathbb{R}^m$. Note from (2.1) and the cartesian nature assumed on \mathbb{C} that $K_i(x^{-i}) = \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\} = \{y_i \in \mathbb{R}^{m_i} \mid y_i \in \mathbb{C}_i, x_j \in \mathbb{C}_j, j \neq i\}$, which is nonempty if $x_j \in \mathbb{C}_j, \forall j \neq i$. Thus $K(x) = \prod K_i(x^{-i}) \neq \emptyset$ if and only if $x \in \mathbb{C}$. Similarly, for $x \in \mathbb{C}$, we have $y \in K(x)$ if and only if $y \in \mathbb{C}$. Therefore $K(x) = \mathbb{C}$ if and only if $x \in \mathbb{C}$.
2. Let $x \in K(x)$ implying that $x_i \in K_i(x^{-i}), \forall i \in \mathcal{N}$, and therefore $(x_i, x^{-i}) \in \mathbb{C}, \forall i \in \mathcal{N}$ and $x \in \mathbb{C}$. The converse follows by noting that $x \in \mathbb{C}$ is equivalent to $(x_i, x^{-i}) \in \mathbb{C} \forall i$, i.e. $x_i \in K_i(x^{-i}), \forall i$ and therefore $x \in K(x)$.
3. Let $x \in \text{dom}(K)$ and $y, z \in K(x)$, i.e. for each $i \in \mathcal{N}$, (y_i, x^{-i}) and $(z_i, x^{-i}) \in \mathbb{C}$. The convexity of $K(x)$ follows by noting that since \mathbb{C} is convex, $((\alpha y_i + (1 - \alpha)z_i), x^{-i}) \in \mathbb{C}$ for each i and $\alpha \in [0, 1]$.
To show closedness, consider a sequence $\{y^k\} \subseteq K(x)$ with limit point \bar{y} . By closedness of \mathbb{C} , for each i , the sequence $\{(y_i^k, x^{-i})\} \subseteq \mathbb{C}$ and $\lim(y_i^k, x^{-i}) = (\bar{y}_i, x^{-i}) \in \mathbb{C}$. Thus $K(x)$ is closed.
4. Suppose x is a point in \mathbb{C} , $d \in K(x)_\infty$ is an arbitrary recession direction and τ is an arbitrary nonnegative number. By convexity of \mathbb{C} , it suffices to show that $x + \tau d \in \mathbb{C}$. Since $d \in K(x)_\infty$, and $x \in K(x)$, the point $x + N\tau d$ belongs to $K(x)$. Therefore the points $z^i := (x_i + N\tau d_i, x^{-i}), i \in \mathcal{N}$, belong to \mathbb{C} . By convexity of \mathbb{C} , the average of these points

$$\frac{1}{N} \sum_{i \in \mathcal{N}} z^i = \frac{N-1}{N} x + \frac{1}{N} (x + N\tau d) = x + \tau d,$$

also belongs to \mathbb{C} , as required. If \mathbb{C} is bounded, $\mathbb{C}_\infty = \{0\}$ and we get $K(x)_\infty \subseteq \{0\}$. Therefore $K(x)_\infty = \{0\}$, implying that $K(x)$ is bounded. ■

■ 2.6.4 Proof of Lemma 2.3

Proof : Suppose $x \in \text{int}(\mathbb{C})$. Then there exist open sets $\mathcal{O}_i \subseteq \mathbb{R}^{m_i}$ containing x_i such that $x \in \mathcal{O} := \prod_{i \in \mathcal{N}} \mathcal{O}_i \subseteq \mathbb{C}$. Then $(\mathcal{O}_i, x^{-i}) := \cup_{y_i \in \mathcal{O}_i} (y_i, x^{-i}) \subseteq \mathbb{C}$, so that $\mathcal{O}_i \subseteq K_i(x^{-i})$, for each

Chapter 2. Refinement of the Generalized Nash Equilibrium

$i \in \mathcal{N}$. It follows that $\mathcal{O} \subseteq K(x)$.

For the converse, let $\text{int}(K(x))$ be nonempty and $x \in \text{int}(K(x))$. Then for each $i \in \mathcal{N}$, x_i belongs to the interior of $K_i(x^{-i})$ (where $K_i(x^{-i})$ is considered a set in \mathbb{R}^{m_i}). Thus there exist open sets $\mathbb{R}^{m_i} \supseteq \mathcal{O}_i \subseteq K_i(x^{-i})$ containing x_i for all i . It follows that $(\mathcal{O}_i, x^{-i}) \subseteq \mathbb{C}$ for all $i \in \mathcal{N}$. Now since \mathbb{C} is convex, the average of these sets is contained in \mathbb{C} , i.e.

$$\mathcal{A} := \sum_{i \in \mathcal{N}} \frac{(\mathcal{O}_i, x^{-i})}{N} = \frac{1}{N} \prod_{i \in \mathcal{N}} \mathcal{O}_i + \frac{N-1}{N} x \subseteq \mathbb{C}.$$

Since $x_i \in \mathcal{O}_i$, \mathcal{A} contains x . Furthermore, \mathcal{A} is open, implying that $x \in \text{int}(\mathbb{C})$. ■

Chapter 3

Global Equilibria of Multi-leader Multi-follower Games

■ 3.1 Introduction

EPECs, or **E**quilibrium **P**rograms with **E**quilibrium **C**onstraints, are a recently explored class of mathematical programs that have caught the interest of researchers in the field of mathematical programming. EPECs arise rather naturally through game theory, out of the modeling of multi-leader multi-follower games. In multi-leader multi-follower games, players are categorized into leaders and followers. Followers compete amongst each other in a conventional Nash game, assuming the decisions of leaders as fixed while leaders compete against each other subject to an equilibrium amongst the followers. The latter can be articulated as an *equilibrium constraint* in the optimization problems of the leaders whereby the equilibrium amongst leaders is the solution of an equilibrium program with equilibrium constraints. Naturally, understanding the equilibria of the EPEC is of great relevance to comprehending the nature of competition between multiple firms faced with multiple followers.

This class of games has gained significant exposure in the last decade particularly through its use when analyzing strategic behavior of firms competing in power markets [HMP00, CHH97, YAO08]. Sherali [She84] examined the existence of an equilibrium in a forward market when the firms are characterized by identical linear costs. Su [Su07] extended this analysis to the non-identical setting while DeMiguel and Xu examined a stochastic counterpart of such games in [DX09]. An “existence” result may be obtained for the weaker notion of a *local Nash equilibrium*, which refers

to the solution of the aggregated stationarity conditions. Ralph and Hu [HR07] appear to have been amongst the first to use such a notion while more recent work by Pang [Pan10] examines when such an equilibrium exists in the context of EPECs. In fact, the *stationarity* approach has been the basis of computational approaches [LM05, Su05]. Finally we note that EPECs may also arise in games other than the multi-leader multi-follower variety, such as Nash games in which each player solves a bilevel optimization. Here, the solution of a lower level optimization problem is posed as an equilibrium constraint in each player's optimization problem.

Before proceeding further, it is worth emphasizing that our work focuses on developing a framework for providing existence statements for *global* equilibria, rather than *local*. Let $\mathcal{N} = \{1, 2, \dots, N\}$ be a set of leaders. In the canonical EPEC, leader i solves a parameterized mathematical program with equilibrium constraints (MPEC) of the following kind.

$$\begin{array}{ll}
 L_i(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \\
 & x_i \in X_i, \\
 & \text{subject to} \quad y_i \in Y_i, \\
 & y_i \in \text{SOL}(F(x_i, x^{-i}, \cdot), K(x_i, x^{-i})).
 \end{array}$$

Here $x_i \in \mathbb{R}^{m_i}$ is leader i 's strategy and y_i is the tuple of follower strategies that form the equilibrium of a Nash game parametrized by the tuple $x = (x_1, \dots, x_N)$, of the strategies of the N leaders. We use the usual notation

$$x^{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \quad \text{and} \quad (\bar{x}_i, x^{-i}) = (x_1, \dots, x_{i-1}, \bar{x}_i, x_{i+1}, \dots, x_N).$$

We assume throughout that X_i, Y_i are closed convex sets. For each x , follower equilibria are characterized by the solution of the variational inequality $\text{VI}(F(x, \cdot), K(x))$ and are denoted as $\text{SOL}(F(x_i, x^{-i}, \cdot), K(x_i, x^{-i}))$. Henceforth we abbreviate

$$\mathcal{S}(x) = \text{SOL}(F(x_i, x^{-i}, \cdot), K(x_i, x^{-i})). \tag{3.1}$$

Though y_i is not within leader i 's control, a minimization over y_i is performed by a leader who is assumed to be *optimistic*, rather than a *pessimistic* leader who would maximize over y_i while

3.1 Introduction

minimizing over x_i . Let $y = (y_1, \dots, y_N)$ and let \mathbb{R}^n be space of (x, y) . By the *feasible region* of the EPEC we mean the set

$$\mathcal{F} = \left\{ (x, y) \in \mathbb{R}^n \left| \begin{array}{l} x_i \in X_i, \\ y_i \in Y_i, \\ y_i \in \mathcal{S}(x), \quad i = 1, \dots, N. \end{array} \right. \right\}. \quad (3.2)$$

We denote this EPEC by \mathcal{E} , and by $\Omega_i(x^{-i}, y^{-i})$, the feasible region of $L_i(x^{-i}, y^{-i})$. It is easily seen that \mathcal{F} is the set of fixed points of

$$\Omega := \prod_{i=1}^N \Omega_i. \quad (3.3)$$

Each objective function φ_i is assumed to be continuous and defined over the range of the set-valued map Ω_i . By a *leader-follower Nash equilibrium*, or simply equilibrium, of this game is meant a tuple of leader strategies $(x, y) \in \mathcal{F}$ such that

$$\varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(u_i, v_i; x^{-i}) \quad \forall (u_i, v_i) \in \Omega_i(x^{-i}, y^{-i}), \quad i = 1, \dots, N.$$

In other words, at the equilibrium, there is no incentive for unilateral deviation for any leader, and given the strategies of the leaders, the followers are at Nash equilibrium. We refer to such an equilibrium as the “global equilibrium,” as opposed to a “local equilibrium.” The latter refers to a tuple (x, y) that satisfies necessary conditions for optimality of each leader’s optimization problem.

Apart from its importance in modeling, the EPEC is also of theoretical interest. The central theoretical question pertaining to EPECs is that of the existence of an equilibrium. There are many *instances* of EPECs for which equilibria have been shown to exist, (for example [DX09, SSM83, Su07]), but it is not clear how these examples may be generalized to a wider class of EPECs. On the other hand, existence results that are theoretically broader are either applicable to EPECs that are essentially single stage games – where there is unique follower equilibrium for each tuple of leader strategies – or they pertain to weaker equilibrium concepts such as stationarity. At the heart of this state of affairs lies the fact that it is not known on what broad mathematical *principle*, the existence of an equilibrium of an EPEC rests. In conventional Nash games with convex strategy sets, such

principles are well known – the fixed point theorems of Brouwer and Kakutani (see [BO99]). Indeed when the feasible region of the EPEC is convex and compact, the EPEC is either a conventional Nash game or a generalized Nash game and the existence of a global equilibrium follows from classical results. But since convexity rarely holds for the equilibrium constraints and due to the presence of the opponent’s strategies in these constraints, these theorems do not apply to EPECs directly. Consequently, while the need to analyze equilibria of EPECs is imminent, there currently exists no paradigm for this purpose that can be expressed and understood in terms of mathematical concepts.

■ 3.1.1 Contributions

This chapter develops such a paradigm for EPECs with *shared constraints*. In such a game, the strategy sets of players are a function of the strategies of their opponents in such a way that there is a *common set*, say \mathbb{C} , lying in the product space of the strategies, that the tuple of player’s strategies are required to lie in. The objective functions, say $f_1, \dots, f_N : \mathbb{C} \rightarrow \mathbb{R}$, are defined only over this common constraint and an equilibrium of this game is a point $(z_1, \dots, z_N) \in \mathbb{C}$ such that for each i

$$f_i(z_1, \dots, z_N) \leq f_i(z_1, \dots, \bar{z}_i, \dots, z_N) \quad \forall \bar{z}_i \quad \text{s.t.} \quad (z_1, \dots, \bar{z}_i, \dots, z_N) \in \mathbb{C}.$$

An extensive introduction to these games is covered in Section 1.1. The EPEC bears a strong resemblance to the shared constraint game because each leader’s decisions are constrained by the equilibrium amongst the same set of followers. However, it turns out, the conventional model of the EPEC, in which leaders solve L_i for $i \in \mathcal{N}$, does not result in a shared constraint game. This work presents *modifications* of this model that result in an EPEC that is a shared-constraint game and lead to a slightly different models, that could be used to model multi-leader multi-follower competition.

The key benefit of converting the EPEC to a shared-constraint game lies in its analysis, for it remedies many of the analytical difficulties that arise in showing the existence of an equilibrium of the EPEC. To elaborate, recall from [BO99] that the existence of an equilibrium of a conventional Nash game can be shown to be equivalent to the existence of a fixed point of a certain mapping,

3.1 Introduction

called the *reaction map*, that maps the strategy space of the game to itself. For a given tuple of strategies x in the domain, the values mapped by the reaction map are the best responses of players, obtained assuming that for each player i the strategies of the opponents are held fixed at x^{-i} . For \mathcal{E} , the map

$$\mathcal{R}(x, y) = (\mathcal{R}_1(x^{-1}, y^{-1}), \times \cdots \times, \mathcal{R}_N(x^{-N}, y^{-N})), \quad \mathcal{R}_i(x^{-i}, y^{-i}) = \text{SOL}(\mathcal{L}_i(x^{-i}, y^{-i})),$$

constitutes the reaction map. Existence of an equilibrium could be claimed by showing that \mathcal{R} admits a fixed point. But if one attempts to apply fixed point theory to \mathcal{R} , several difficulties emerge. Almost all fixed point theorems rely on three assumptions: (a) the mapping to which a fixed point is sought is assumed to be a *self-mapping*; (i.e. it maps its domain to itself, or subsets of itself) (b) the mapping is required to be continuous (if the mapping is single-valued) or upper semicontinuous (if set-valued); and (c) the domain of the mapping and the mapped values are required to be of a specific shape, e.g. convex. The first difficulty one encounters is that this reaction map is not necessarily a self-mapping, since $\text{range}(\Omega)$ may not be a subset of $\text{dom}(\Omega)$. Secondly, the continuity (or upper semicontinuity) of \mathcal{R} is far from immediate and usually requires Ω to be continuous, which is perhaps the biggest difficulty of all. And finally, $\text{dom}(\Omega)$ is hard to characterize and little can be said about its shape.

These three difficulties listed are germane to any coupled constraint game, but they are more pronounced in the EPEC since continuity of \mathcal{S} is hard to guarantee, except for some trivial cases. The analytically powerful feature of shared-constraint games is that another map can be constructed whose fixed points are the equilibria of the shared-constraint game. This map is a self-mapping, it is upper semicontinuous under mild assumptions, and its domain is \mathcal{F} , which is much easier to characterize. Therefore, in applying fixed point theory to this map, the challenge lies in dealing with the nonconvexity of \mathcal{F} . A formal approach to handling nonconvexity of equilibrium constraints based on topological fixed point theory and the theory of retracts is used to overcome this difficulty. We also observe that if the objective functions are defined over the entire space, the use of the theory of retracts can be avoided; this is discussed in Section 3.4.1.

The theoretical message of our work is that (a) a shared-constraint formulation of multi-leader multi-follower games has much more analytical tractability and (b) *contractibility*, as opposed to

convexity, is a key property for the existence of equilibria to EPECs obtained in this way. Since topological fixed point theory is very general and contains the convex fixed point theorems of Brouwer and Kakutani, what results is a very general, and to a great extent, unifying theory of the existence of equilibria, that contains the already known theory of equilibria for convex constrained games. We are not the first to notice this. Debreu [Deb52], Tsefatsion [Tes83] and McClendon [McC86] have applied topological fixed point theory to abstract games with general strategy sets. Our contribution is in posing shared-constraint games as alternative formulations for multi-leader multi-follower competition and showing that the analysis of the resulting EPECs can be accomplished through topological fixed point theory. Our work leads to a broad existence theory for these EPECs and the identification of open problems, mostly relating to the nature of solution sets of MPECs, which if answered, would broaden this theory.

A note is due of the assumptions we make and the kind of results we derive. Our intention in this chapter is to present a foundational theory for existence of equilibria of EPECs with shared constraints. Consequently, we derive results that are valuable for both their generality and their explanatory power and thereby allow for understanding more precisely the conceptual barriers to stronger results. Such barriers may take the form of an inability to immediately verify a required sufficiency condition; they are not addressed in this thesis. Most results are followed by a discussions on the reach of the result and possible ways of avoiding pathology so as to widen this reach. One of our foci is also that when specialized to conventional Nash games with convex constraints, our assumptions and results emerge as equivalent to known results for such games. Indeed one of our motivations for taking the route of topological fixed point theory is this unification. Finally, note that in attempting to develop results of this kind, we are unable to provide the sharper conclusions that are common in works where specific EPECs are considered and structural assumptions are made.

The chapter is organized as follows. Section 3.2 introduces the EPEC associated with a canonical multi-leader multi-follower game and presents modifications of this model for constructing a shared-constraint game. We discuss how fixed point theory may be applied to the latter through the modified reaction map. Furthermore we discuss relations between these modifications and their relation to the original model. In Section 3.3, we present a review of the nonconvex fixed point theory that is used in this work. Section 3.4 applies this fixed point theory to the modified reaction

3.2 The EPEC as a shared-constraint game

map derived in Section 3.2 while in Section 3.5, we consider some instances of EPECs and examine the implications of the results from Section 3.4 to these EPECs. We conclude in Section 3.6 with a brief summary.

■ 3.2 The EPEC as a shared-constraint game

We begin with the model from the introduction wherein player i then solves the MPEC

$$\begin{array}{ll}
 L_i(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \\
 & x_i \in X_i, \\
 & \text{subject to} \quad y_i \in Y_i, \\
 & y_i \in \mathcal{S}(x).
 \end{array}$$

The logic behind using the follower decision y_i as a decision variable of the leader problem is that the leader is taking his decisions in an *optimistic* manner; specifically he optimizes over the set of all values that follower equilibria could yield. Two chief distinctions arise in EPEC solution concepts when one has to decide the nature of follower decisions *at equilibrium*. Since the optimal y_i 's are, after all, conjectures on the part of the leader, one school of thought allows these y_i 's to be distinct across leaders. The leader follower equilibrium defined above pertains to this thinking. Another school of thought requires that these conjectures be *consistent*, i.e. at equilibrium $y_i = y_j$, for all $i, j \in \mathcal{N}$. This would be relevant if the equilibrium is to be interpreted in practical settings.

The goal of this section is to reformulate the EPEC as a shared-constraint game and present the analytical consequences of this reformulation. Section 3.2.1 gives a background of coupled constraint games. Section 3.2.2 presents the shared-constraint reformulations and Section 3.2.3 contains the analytical consequences. The analytical consequences include showing that a sufficient condition for the existence of an equilibrium to an EPEC is the existence of fixed point to a certain *modified reaction map*. Section 3.2.5 presents a comparison of the equilibria of the modifications with that of the original EPEC.

■ 3.2.1 Coupled constraint games

A game is defined by a set of players, a strategy space for each player, a set of permissible strategies and real valued objective functions defined on the set of permissible strategies. The set of permissible strategies is a subset of the product space formed from taking the cartesian product of the strategy spaces of all players. All tuples of player strategies are required to lie in the set of permissible strategies. In classical Nash games, the set of permissible strategies is itself a cartesian product of sets drawn from strategy spaces.

The EPEC is a highly nonconvex *coupled constraint game* with not necessarily *shared constraints*. A game is said to be a *coupled constraint game* if the strategies available to a player depend on the the strategies chosen by his opponents. A game is said to be a *coupled constraint game* if the feasible region of each player's optimization problem is a function of the strategies of other players. The feasible region mapping $\Omega = \prod_{i=1}^N \Omega_i$ defined in (3.3) (where $\Omega_i(x^{-i}, y^{-i})$ is the set of (x_i, y_i) feasible for $L_i(x^{-i}, y^{-i})$) is said to be a *shared constraint* if Ω has the following structure: for (x, y) in the domain of Ω ,

$$(u, v) \in \Omega(x, y) \iff (u_i, x^{-i}, v_i, y^{-i}) \in \mathcal{F} \quad \forall i. \quad (3.4)$$

Such an Ω is completely defined by its fixed point set, \mathcal{F} . Shared constraint games arise naturally when players face a *common constraint*, e.g. in a bandwidth sharing game, and are an area of flourishing recent research; see [FK07].

Shared constraint games were introduced by Rosen [Ros65] as a rigorous generalization of the classical Nash game. A set $\mathbb{C} \subseteq \mathbb{R}^n$ is taken as the set of permissible strategies and objective functions f_i are defined as $\mathbb{C} \rightarrow \mathbb{R}$ mappings. The equilibrium of this game is a point $z = (z_1, \dots, z_N)$ such that

$$z \in \mathbb{C}, \quad f_i(z_1, \dots, z_N) \leq f_i(z_1, \dots, \bar{z}_i, \dots, z_N) \quad \forall \bar{z}_i \text{ s.t. } (z_1, \dots, \bar{z}_i, \dots, z_N) \in \mathbb{C}, \quad \forall i. \quad (3.5)$$

This definition is sound since f_i 's are defined over \mathbb{C} . It is also consistent with the idea that each player have no incentive for unilateral deviation over the space of permissible strategies. An alternative, equivalent definition, introduced by Harker [Har91]: z is an equilibrium if $z \in \Omega^{\mathbb{C}}(z)$

3.2 The EPEC as a shared-constraint game

and for all i

$$f_i(z_1, \dots, z_N) \leq f_i(z_1, \dots, \bar{z}_i, \dots, z_N) \quad \forall \bar{z}_i \in \Omega_i^{\mathbb{C}}(z^{-i}),$$

where

$$\Omega^{\mathbb{C}} = \prod_{i=1}^N \Omega_i^{\mathbb{C}}, \quad \text{and} \quad \Omega_i^{\mathbb{C}}(z^{-i}) = \{\bar{z}_i \mid (\bar{z}_i; z^{-i}) \in \mathbb{C}\}.$$

These games have subsequently been extended to coupled constraints games with *not necessarily* shared constraints. In such a game, an equilibrium is a point z such that

$$z \in \prod_{i=1}^N \Omega_i^{NS}(z^{-i}), \quad f_i(z_1, \dots, z_N) \leq f_i(z_1, \dots, \bar{z}_i, \dots, z_N) \quad \forall \bar{z}_i \in \Omega_i^{NS}(z^{-i}), \quad \forall i.$$

Here Ω_i^{NS} is *any* convex valued set-valued map, not necessarily of the form of a shared constraint.

If we let \mathbb{C}_i to be the graph of Ω_i^{NS} , we may rewrite the above as

$$z \in \bigcap_{i=1}^N \mathbb{C}_i, \quad f_i(z_1, \dots, z_N) \leq f_i(z_1, \dots, \bar{z}_i, \dots, z_N) \quad \forall (\bar{z}_i, z^{-i}) \in \mathbb{C}_i \quad \forall i. \quad (3.6)$$

Comparing (3.6) with (3.5) reveals that the coupled constraint game without shared constraints has some features distinct from conventional Nash game and shared-constraint games. The space of permissible strategies is different for each player and at equilibrium, each player i has no incentive for unilateral deviation over *his own space* of permissible strategies, \mathbb{C}_i , which is also the domain of f_i . The equilibrium is thus a point that lies in the intersection of all spaces of permissible strategies, $\cap \mathbb{C}_i$. This leaves the possibility that $\cap \mathbb{C}_i$ could be empty, even while \mathbb{C}_i are nonempty. This is impossible for a shared constraint, since the graph of $\Omega_i^{\mathbb{C}}$ is \mathbb{C} for all i . Indeed Arrow and Debreu in [AD54] recognize that such games are technically not *games* and instead call them *abstract economies*.

Less is known in literature about coupled constraint games without shared constraint even with convex constraints. This difficulty is intensified in the EPEC due to the inherent lack of regularity of the constraints involved. On the contrary, much has been said about shared-constraint games, with convex fixed point sets (See particularly, the works of Rosen [Ros65], Facchinei and Pang [FP09], Facchinei et al. [FFP07] and Chapter 2 in this thesis). It is tempting to therefore think that in the case where the Ω is a shared constraint, the EPEC will be more amenable and a more complete

understanding could be obtained for it.

■ 3.2.2 Shared constraint formulations

The setting that motivates the EPEC has a strong and natural resemblance to shared-constraint games: each leader in a multi-leader-multi-follower game is constrained by the equilibrium amongst the same set of followers. However this constraint is not automatically a shared-constraint because the equilibrium of the followers is not a constant, but a variable of each leader's decision problem. The possible disparity in the conjectures that leaders make about follower equilibria results in this constraint not being shared. Even definitions that require these conjectures to be consistent, only require this consistency *at equilibrium*; consistency is not required to prevail in individual optimization problems and is therefore not enough to make the constraint shared¹. Note that the single-valuedness of \mathcal{S} also enforces consistency only at equilibrium, not in the leader's optimization problems.

The objective of this section is to “make” the EPEC a shared-constraint game. We demonstrate other formulations of the EPEC that result in shared-constraint games. The analytical advantages of having such an EPEC will be seen in Section 3.2.3. Note that the goal of this work is the study of EPECs and to present an analytical theory for them; understanding the game theoretic meaning of the formulations and modifications that follow is beyond the scope of this work.

Leaders sharing all equilibrium constraints: Consider the formulation in which the i^{th} leader solves the following optimization problem.

$$\begin{array}{ll}
 L_i^{ae}(x^{-i}, y^{-i}) & \text{minimize}_{x_i, y_i} \quad \varphi_i(x_i, y_i; x^{-i}) \\
 & x_i \in X_i, \\
 & \text{subject to} \quad y_i \in Y_i, \\
 & y_j \in \mathcal{S}(x) \quad j = 1, \dots, N.
 \end{array}$$

¹Indeed this is a common feature of many “equilibrium” definitions for the EPECs, wherein *at equilibrium* more requirements are imposed on the solutions of the players' optimization problems, which do not necessarily hold for solutions that are not equilibria.

3.2 The EPEC as a shared-constraint game

Let this game be denoted by \mathcal{E}^{ae} . The difference between this problem and L_i is that *all* constraints

$$y_j \in \mathcal{S}(x), \quad j = 1, \dots, N$$

are now a part of *each* leader's optimization problem. As a consequence, while y_i satisfies the same constraints as in $L_i(x^{-i}, y^{-i})$, x_i is constrained by additional constraints. For $y_j \in Y_j, x_j \in X_j$ for $j \neq i$, let $\Omega_i^{ae}(x^{-i}, y^{-i})$ be the feasible region of $L_i^{ae}(x^{-i}, y^{-i})$. i.e.

$$\Omega_i^{ae}(x^{-i}, y^{-i}) = \{x_i, y_i \mid x_i \in X_i, y_i \in Y_i, y \in \mathcal{S}^N(x)\} = \{x_i, y_i \mid x \in X, y \in Y, (x, y) \in \mathcal{G}\},$$

where

$$Y = \prod_{i=1}^N Y_i, \quad X = \prod_{i=1}^N X_i, \quad \mathcal{S}^N = \prod_{i=1}^N \mathcal{S}, \quad \text{and} \quad \mathcal{G} = \{(x, y) \mid y \in \mathcal{S}^N(x)\}, \quad (3.7)$$

is the graph of \mathcal{S}^N . It easy to see that Ω_i^{ae} is in the form dictated by (3.4). Furthermore, the fixed points of $\Omega^{ae} = \prod_{i=1}^N \Omega_i^{ae}$ are the same as that of Ω .

$$(x, y) \in \Omega^{ae}(x, y) \iff (x, y) \in \Omega(x, y) \iff (x, y) \in \mathcal{F} = (X \times Y) \cap \mathcal{G}.$$

An equilibrium of \mathcal{E}^{ae} is a point

$$(x, y) \in \mathcal{F}, \text{ such that } \varphi_i(x_i, y_i; x^{-i}) \leq \varphi_i(\bar{x}_i, \bar{y}_i; x^{-i}) \quad \forall (\bar{x}_i, \bar{y}_i) \in \Omega_i^{ae}(x^{-i}, y^{-i}), \forall i.$$

This immediately leads us to the following result.

Lemma 3.1 *Every equilibrium of \mathcal{E} is an equilibrium of \mathcal{E}^{ae} .*

Proof: An equilibrium (x, y) of \mathcal{E} is a point in \mathcal{F} , and hence feasible for \mathcal{E}^{ae} . Since $\Omega_i^{ae}(x^{-i}, y^{-i}) \subseteq \Omega_i(x^{-i}, y^{-i})$, the result follows. ■

Leaders with consistent conjectures: In EPECs, often a requirement is imposed that at equilibrium the conjectures made by leaders about follower equilibria be consistent, $y_i = y_j$, for all i, j . It is not immediately clear if such consistency should be an exogenous requirement on the game, or if it should be imposed as a part of each leader's optimization problem. If we impose the

Chapter 3. Global Equilibria of Multi-leader Multi-follower Games

consistency requirement endogenously the resulting EPEC, \mathcal{E}^{cc} , turns out to be a shared-constraint game in which the i th leader solves the following problem.

$$\begin{array}{ll}
 L_i^{cc}(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \\
 & x_i \in X_i, \\
 & \text{subject to} \quad y_i \in Y_i, \\
 & y_i \in \mathcal{S}(x) \\
 & y_j = y_1, \quad j = 1, \dots, N,
 \end{array}$$

Let $y_j = y_\ell, y_j \in Y_j, x_j \in X_j$, for all $j, \ell \neq i$. Let $\Omega_i^{cc}(x^{-i}, y^{-i})$ denote the feasible region of $L_i^{cc}(x^{-i}, y^{-i})$

$$\begin{aligned}
 \Omega_i^{cc}(x^{-i}, y^{-i}) &= \{x_i, y_i \mid x_i \in X_i, y_i \in Y_i, y_i \in \mathcal{S}(x), y_j = y_1, j = 1, \dots, N\} \\
 &= \{x_i, y_i \mid x \in X, y \in Y, y \in \mathcal{S}^N(x), y \in \mathbf{A}\}
 \end{aligned}$$

where

$$\mathbf{A} = \{y \mid y_i = y_1, i = 1, \dots, N\}, \quad (3.8)$$

Consequently, $\Omega^{cc} = \prod_{i=1}^N \Omega_i^{cc}$ is a shared constraint. Let \mathcal{F}^{cc} be set of fixed points of Ω^{cc} , given by

$$\mathcal{F}^{cc} = (X \times (Y \cap \mathbf{A})) \cap \mathcal{G}.$$

Lemma 3.2 *If \mathcal{S} is single-valued, we have $\mathcal{F} = \mathcal{F}^{cc}$.*

Proof : It is obvious that $\mathcal{F}^{cc} \subseteq \mathcal{F}$. To show the claim, let $(x, y) \in \mathcal{F}$. Since \mathcal{S} is single valued, $y \in \mathbf{A}$ and consequently $(x, y) \in \mathcal{F}^{cc}$. ■

Lemma 3.3 *Let \mathcal{S} be single-valued. Every equilibrium of \mathcal{E} is an equilibrium of \mathcal{E}^{cc} .*

Proof : An equilibrium (x, y) of \mathcal{E} is in \mathcal{F} , and so by Lemma 3.2, $(x, y) \in \mathcal{F}^{cc}$. Since $\Omega_i^{cc}(x^{-i}, y^{-i}) \subseteq \Omega_i(x^{-i}, y^{-i})$, the result follows. ■

3.2 The EPEC as a shared-constraint game

Notice that \mathcal{E}^{ae} retains equilibria of \mathcal{E} even when \mathcal{S} is set-valued, while we are able to claim a similar property for \mathcal{E}^{cc} only under the single-valuedness of \mathcal{S} .

Remark : We wish to emphasize the following facts:

- That a constraint is shared does not imply that the conjectures are consistent.
- That conjectures are consistent *at equilibrium*, (say if \mathcal{S} is single-valued) does not imply that constraint is shared.
- If the conjectures are consistent in individual optimization problems, they are consistent at equilibrium and the constraint is shared. \square

Leaders solving bilevel optimization problems: The game in which leaders solve bilevel optimization problems *without* coupled constraints is also an EPEC with shared constraints. Consider a game where the i^{th} leader solves

$L_i^{bl}(x^{-i}, y^{-i})$	<div style="text-align: center;"> $\underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i})$ </div> <div style="text-align: center;"> $x_i \in X_i,$ </div> <div style="text-align: center;"> $\text{subject to} \quad y_i \in \widehat{\mathcal{S}}_i(x_i),$ </div> <div style="text-align: center;"> $y_i \in Y_i.$ </div>
----------------------------	--

Notice the absence of the coupling of leader decisions in the constraints. Let Ω_i^{bl} be the feasible region of L_i^{bl} and let \mathcal{F}^{bl} be the set of fixed points of $\Omega^{bl} := \prod_{i=1}^N \Omega_i^{bl}$. Since there is no coupling, it is easily seen that

$$\mathcal{F}^{bl} = \Omega^{bl} = \{(x, y) \mid x \in X, y \in Y, (x, y) \in \widehat{\mathcal{G}}\},$$

where $\widehat{\mathcal{G}} = \prod_{i=1}^N \widehat{\mathcal{G}}_i$ and $\widehat{\mathcal{G}}_i$ is the graph of $\widehat{\mathcal{S}}_i$. If $(x_j, y_j) \in \Omega_j^{bl}$, for $j \neq i$,

$$\Omega_i^{bl} = \{(x_i, y_i) \mid x_i, y_i\} = \{(x_i, y_i) \mid (x, y) \in \mathcal{F}^{bl}\}.$$

Clearly Ω^{bl} is a shared constraint.

Leaders with objectives independent of follower equilibrium: This game, though not a shared constraint game, obeys a key result that holds for shared-constraint games. Here we assume that the i^{th} leader solves the following problem.

$$\begin{array}{ll}
 L_i^{\text{ind}}(x^{-i}, y^{-i}) & \text{minimize}_{x_i, y_i} \quad \varphi_i(x_i; x^{-i}) \\
 & x_i \in X_i, \\
 & \text{subject to} \quad y_i \in Y_i, \\
 & y_i \in \mathcal{S}(x).
 \end{array}$$

We denote this game by \mathcal{E}^{ind} and discuss it at the end of Section 3.2.3.

■ 3.2.3 Fixed point formulation through the modified reaction map

At the classical Nash equilibrium, each player's strategy is his "best response" assuming the strategies of his opponents are held fixed. For any tuple of strategies $x = x_1, \dots, x_N$, one may obtain a tuple of "best responses" in the following manner: take the i^{th} "element" of the tuple to be the solutions of player i 's optimization problem obtained from assuming the opponent's strategies fixed at x^{-i} . The resulting best response would, in general, be a set-valued function of x , mapping the space of strategies to subsets of this space. This function is often called the *reaction map*. The Nash equilibrium is a fixed point of this map.

When the feasible region of each player's optimization problem is convex and independent of his opponent's strategies, and each player's objective function is convex in his own strategy, this map is upper semicontinuous and convex-valued. Thus if the space of strategies of players, which forms the domain of the reaction map, is also compact, Kakutani's fixed point theorem yields the existence of the fixed point to the reaction map, i.e. a Nash equilibrium exists.

The nonconvexity of an EPEC implies that there is no simple characterization of optimality, and since our interest is in global equilibria, a return to first principles, as above, is the only way ahead. However, when the strategy set of a player is dependent on the strategies of his opponents, difficulties arise in the above modus operandi when one attempts to apply fixed point theory to the reaction map of such a game. To illustrate this, let us define the following reaction map for the

3.2 The EPEC as a shared-constraint game

EPEC \mathcal{E} mentioned in Section 3.1. Let $\mathcal{R} : \text{dom}(\Omega) \rightarrow 2^{\text{range}(\Omega)}$,

$$\mathcal{R}(x, y) = \left\{ (\bar{x}, \bar{y}) \in \Omega(x, y) \left| \begin{array}{ccc} \varphi_1(\bar{x}_1, \bar{y}_1; x^{-1}) & \leq & \varphi_1(u_1, v_1; x^{-1}) \\ & \vdots & \\ \varphi_N(\bar{x}_N, \bar{y}_N; x^{-N}) & \leq & \varphi_N(u_N, v_N; x^{-N}) \end{array} \right. \right\} \forall (u, v) \in \Omega(x, y) \quad (3.9)$$

(x, y) is an equilibrium of \mathcal{E} if and only if (x, y) is a fixed point of \mathcal{R} . Almost all fixed point theorems rely on the following broad assumptions:

- (a) the mapping to which a fixed point is sought is assumed to be a *self-mapping*;
- (b) the mapping is required to be continuous (if the mapping is single-valued) or upper semicontinuous (if set-valued);
- (c) the domain of the mapping and the mapped values are required to be of a specific shape, e.g. convex.

The first difficulty one encounters is that \mathcal{R} is not necessarily a self-mapping, since $\text{range}(\Omega)$ may not be a subset of $\text{dom}(\Omega)$ ². Secondly, the continuity (or upper semicontinuity) of \mathcal{R} is far from immediate. This usually requires Ω to be continuous. Finally, $\text{dom}(\Omega)$ is hard to characterize and little can be said about its shape.

A compelling feature of shared-constraint games is that all three of these difficulties can be circumvented through the use of another map whose fixed points are also equilibria of \mathcal{E} . Assume Ω is a shared constraint, with a fixed point set \mathcal{F} . Let $\Psi : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$ be given by

$$\Psi(x, y, \bar{x}, \bar{y}) = \sum_{i=1}^N \varphi_i(\bar{x}_i, \bar{y}_i; x^{-i}) \quad \forall (x, y), (\bar{x}, \bar{y}) \in \mathcal{F}. \quad (3.10)$$

and consider the map *modified reaction map* $\Upsilon : \mathcal{F} \rightarrow 2^{\mathcal{F}}$, defined as

$$\Upsilon(x, y) := \left\{ (\bar{x}, \bar{y}) \in \mathcal{F} \mid \Psi(x, y, \bar{x}, \bar{y}) = \inf_{(u, v) \in \mathcal{F}} \Psi(x, y, u, v) \right\}. \quad (3.11)$$

²Of course, if $\Omega(x, y) \neq \emptyset$ for all $(x, y) \in X \times Y$, we have $\text{dom}(\Omega) = X \times Y$ and \mathcal{R} to be a map from $X \times Y$ to subsets of $X \times Y$. But one of the key difficulties [LPR96] of MPECs and EPECs is that the domain of the problem is defined implicitly through \mathcal{S} . In this work, we seek a theory that does not rely on the verification of the statement “ $\text{dom}(\Omega) = X \times Y$ ”. The approach of Arrow et al. [AD54] assumes this statement.

Chapter 3. Global Equilibria of Multi-leader Multi-follower Games

We show below that a fixed point of Υ is an equilibrium of \mathcal{E} . Notice that the difficulties due to (a) and (b) that arise in \mathcal{R} do not appear Υ : Υ is a self-mapping and because the infimum in (3.11) is over a set that is independent of (x, y) , the upper semicontinuity of Υ is easily obtained. (c) remains a hurdle. However since \mathcal{F} is much easier to characterize than $\text{dom}(\Omega)$, (c) can be approached with greater ease for Υ than for \mathcal{R} . Note that the map Υ is analogous to that used by Rosen [Ros65], Theorem 1. Rosen's work is the inspiration for our approach.

Lemma 3.4 *If Ω is a shared constraint, i.e. it satisfies (3.4), then every fixed point of Υ is a fixed point of \mathcal{R} .*

Proof : We deny the claim and show that it results in a contradiction. Suppose $(x, y) \in \Upsilon(x, y)$ and $(x, y) \notin \mathcal{R}(x, y)$. There exists $(u, v) \in \Omega(x, y)$ and i such that

$$\varphi_i(u_i, v_i; x^{-i}) < \varphi_i(x_i, y_i; x^{-i}).$$

Since $(u, v) \in \Omega(x, y)$, we must have $(u_i, x^{-i}, v_i, y^{-i}) \in \mathcal{F}$. But this means

$$\Psi(x, y, u_i, x^{-i}, v_i, y^{-i}) < \Psi(x, y, x, y),$$

a contradiction to $(x, y) \in \Upsilon(x, y)$. ■

Remark : It is not true that every equilibrium of the EPEC is a fixed point Υ . This can be checked easily by considering a hypothetical case with convex \mathcal{F} , wherein it is well known that fixed points of \mathcal{R} and Υ can be very different, see Chapter 2 where this is discussed in detail. Existence of a fixed point to Υ is only a sufficient condition for an equilibrium of the EPEC to exist. There may exist fixed points of \mathcal{R} that are not fixed points of Υ and games for which there are fixed points to \mathcal{R} , but none to Υ , as shown in Example 2.2.

When \mathcal{F} is convex, Theorem 3.4 and the map Υ also has an interesting connection with the “variational equilibrium” (see Section 1.1.1 and Chapter 2) in games with convex shared constraints. Suppose \mathcal{F} is convex and each φ_i is convex in (x_i, y_i) . Fixed points of \mathcal{R} form the generalized Nash equilibrium [Har91] of the game. The variational equilibrium is defined as the solution of the variational inequality $\text{VI}(\mathcal{F}, F)$, where $F = (\nabla_1 \varphi_1^T, \dots, \nabla_N \varphi_N^T)^T$, and it is easy to see that it is the

3.2 The EPEC as a shared-constraint game

same as the fixed points of Υ . □

Recall the third category of shared-constraint games, \mathcal{E}^{ind} , wherein the objective functions of leaders are independent of the follower equilibrium. We show that fixed points of Υ are equilibria of \mathcal{E}^{ind} . Suppose each leader solves the following MPEC.

$$\begin{array}{ll}
 L_i^{ind}(x^{-i}, y^{-i}) & \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i; x^{-i}) \\
 & x_i \in X_i, \\
 \text{subject to} & y_i \in Y_i, \\
 & y_i \in \mathcal{S}(x).
 \end{array}$$

Notice that y_i is a variable of the above MPEC, though it does not appear explicitly in the objective. We also assume that $Y_i = Y_j$ for all i, j . Fixed points of Υ are also equilibria of this game. Observe that in this case,

$$\Psi(x, y, \bar{x}, \bar{y}) = \sum_{i=1}^N \varphi_i(\bar{x}_i; x^{-i}),$$

is also independent of \bar{y} .

Lemma 3.5 *If (x, y) is a fixed point of Υ , it is an equilibrium of \mathcal{E}^{ind} .*

Proof : Again we prove this by contradiction. Let $(x, y) \in \Upsilon(x, y)$ and $(x, y) \notin \mathcal{R}(x, y)$. So there exists $(u, v) \in \Omega(x, y)$ and leader i such that

$$\begin{aligned}
 & \varphi_i(u_i; x^{-i}) < \varphi_i(x_i; x^{-i}), \\
 & \text{where } u_i \in X_i, \quad v_i \in Y_i, \quad v_i \in \mathcal{S}(u_i; x^{-i}).
 \end{aligned}$$

Let (\bar{x}, \bar{y}) be the point given as

$$\bar{x} = (x_1, \dots, u_i, \dots, x_N) = (u_i; x^{-i}) \quad \bar{y} = (v_i, \dots, v_i).$$

Since $(x, y) \in \mathcal{F}$, $x_j \in X_j$, for all j , whereby $\bar{x} \in X$. Since $Y_i = Y_j$ for all j , $\bar{y} \in Y$ and since

Chapter 3. Global Equilibria of Multi-leader Multi-follower Games

$v_i \in \mathcal{S}(\bar{x})$, it follows that $\bar{y} \in \mathcal{S}^N(\bar{x})$. It follows from this that $(\bar{x}, \bar{y}) \in \mathcal{F}$. So

$$\Psi(x, y, \bar{x}, \bar{y}) = \sum_{j \neq i} \varphi_j(x_j; x^{-j}) + \varphi_i(u_i; x^{-i}) < \Psi(x, y, x, y).$$

This contradicts $(x, y) \in \Upsilon(x, y)$. ■

Remark : Notice that when \mathcal{S} is single-valued, the fixed points of Υ are equilibria that are common to \mathcal{E}^{ae} and \mathcal{E}^{cc} . This is so because of Lemma 3.2, which gives $\mathcal{F}^{cc} = \mathcal{F}$ for single-valued \mathcal{S} , and Lemma 3.4 which shows that fixed points of Υ are equilibria of \mathcal{E}^{cc} and \mathcal{E}^{ae} . □

We now come to an example presented in [PF05] wherein the authors produced an EPEC that had no solution. We observe that this EPEC is not a shared-constraint game. But if this EPEC is modified in the form of \mathcal{E}^{ae} , the modified EPEC does admit an equilibrium.

Example 3.1. The example comprises of 2 leaders and 1 follower. $X_1 = X_2 = [0, 1]$ and $Y = \mathbb{R}$. The lone follower is assumed to solve the optimization problem

$$\min_{y \geq 0} \{y(-1 + x_1 + x_2) + \frac{1}{2}y^2\} = \max\{0, 1 - x_1 - x_2\}$$

The leaders, notably, have objectives independent of the strategies of the other leader and thus solve the following optimization problems.

$\begin{array}{ll} \text{L}_1 & \underset{x_1, y_1}{\text{minimize}} \quad \varphi_1(x_1, y_1; x_2) = \frac{1}{2}x_1 + y_1 \\ & \text{subject to} \quad x_1 \in [0, 1] \\ & \quad \quad \quad y_1 = \max\{0, 1 - x_1 - x_2\} \end{array}$	$\begin{array}{ll} \text{L}_2 & \underset{x_2, y_2}{\text{minimize}} \quad \varphi_2(x_2, y_2; x_1) = -\frac{1}{2}x_2 - y_2 \\ & \text{subject to} \quad x_2 \in [0, 1] \\ & \quad \quad \quad y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}$
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One may explicitly substitute the optimal y_1, y_2 in terms of x_1, x_2 to obtain a reaction map in the

3.2 The EPEC as a shared-constraint game

x_1, x_2 space.

$$\mathcal{R}_1(x_2) = \{1 - x_2\} \quad \forall x_2 \in [0, 1] \quad \mathcal{R}_2(x_1) = \begin{cases} \{0\} & x_1 \in [0, \frac{1}{2}) \\ \{0, 1\} & x_1 = \frac{1}{2} \\ \{1\} & x_2 \in (\frac{1}{2}, 0]. \end{cases}$$

It is easy to see that this map has no fixed point, whereby this game has no equilibrium.

Now consider the following problem in which both leaders see both equilibrium constraints.

$\begin{aligned} \text{L}_1 \quad & \underset{x_1, y_1}{\text{minimize}} \quad \varphi_1(x_1, y_1; x_2) = \frac{1}{2}x_1 + y_1 \\ & x_1 \in [0, 1] \\ \text{subject to} \quad & y_1 = \max\{0, 1 - x_1 - x_2\} \\ & y_2 = \max\{0, 1 - x_1 - x_2\} \end{aligned}$	$\begin{aligned} \text{L}_2 \quad & \underset{x_2, y_2}{\text{minimize}} \quad \varphi_2(x_2, y_2; x_1) = -\frac{1}{2}x_2 - y_2 \\ & x_2 \in [0, 1] \\ \text{subject to} \quad & y_1 = \max\{0, 1 - x_1 - x_2\} \\ & y_2 = \max\{0, 1 - x_1 - x_2\} \end{aligned}$
---	--

This is clearly a shared-constraint game with $\mathcal{F} = \{(x_1, x_2) \in [0, 1]^2, (y_1, y_2) \geq 0 \mid y_1 = \max\{0, 1 - x_1 - x_2\} = y_2\}$. We have

$$\Psi(x, y, \bar{x}, \bar{y}) = \frac{1}{2}\bar{x}_1 + \bar{y}_1 - \frac{1}{2}\bar{x}_2 - \bar{y}_2,$$

which, importantly, is independent of (x, y) . So

$$\begin{aligned} \Upsilon(x, y) &= \arg \min_{(\bar{x}, \bar{y}) \in \mathcal{F}} \Psi(x, y, \bar{x}, \bar{y}) \\ &= \arg \min_{(\bar{x}, \bar{y}) \in \mathcal{F}} \frac{1}{2}\bar{x}_1 - \frac{1}{2}\bar{x}_2 \\ &= ((0, 1), (0, 0)). \end{aligned}$$

It follows thereby that $((x_1, x_2), (y_1, y_2)) = ((0, 1), (0, 0))$ is an equilibrium of the modified game. It is illuminating to study the objectives of the two at equilibrium. Leader 1 gets $\varphi_1(0, 0, 1) = 0$ whereas leader 2 gets $\varphi_2(1, 0, 0) = -\frac{1}{2}$. 0 is leader 1's global minimum and he thus has no incentive to deviate from it. For leader 2, the strategy set at equilibrium reduces to a singleton containing only his equilibrium strategy. This is induced by the presence of leader 1's equilibrium constraint in his optimization problem. To see this notice that the constraint $y_1 = \max\{0, 1 - x_1, x_2\}$ is, at

Chapter 3. Global Equilibria of Multi-leader Multi-follower Games

equilibrium, equivalent to $0 = \max\{0, 1 - x_2\}$. This, together with the constraint $x_2 \in [0, 1]$ implies $x_2 = 1$ and consequently $y_2 = 0$. \square

Remark : The above example has several messages that we can learn from, but we make some cautionary remarks so that we do not convey the wrong message.

- All EPECs when modified to a shared-constraint form may not admit equilibria. The key feature of the above example that enables an equilibrium is that the objectives of leaders are independent of the strategies of other leaders. For any such game Υ is a constant map, and thus admits a fixed point. Of course the independence of φ_i from x^{-i} is not necessary to obtain a constant-valued Υ . In the above example we may take

$$\varphi_1(x_1, y_1; x_2) = \frac{1}{2}x_1 + y_1 + f_1(x_2), \quad \varphi_2(x_2, y_2; x_1) = -\frac{1}{2}x_2 - y_2 + f_2(x_1),$$

where f_1, f_2 are any continuous functions. Because of the separability in the objective functions, we would still get that Υ is a constant map:

$$\Upsilon(x, y) = \arg \min_{(\bar{x}, \bar{y}) \in \mathcal{F}} \frac{1}{2}\bar{x}_1 - \frac{1}{2}\bar{x}_2 + f_1(x_2) + f_2(x_1) = ((0, 1), (0, 0)).$$

- Note also that shared-constraint formulations are merely alternative formulations of multi-leader multi-follower games that are analytically more tractable than the original formulation. We do not mean to suggest that merely their tractability, as opposed to their meaningfulness, be used as the basis for their applicability.
- A deeper study is needed to comprehensively understand the applicability of shared-constraint formulations and to assess if there exist other formulations that result in EPECs with shared constraints. \square

We formalize these remarks through the following theorem, which also is our first existence result.

Theorem 3.6 *Let Ω be a shared constraint as defined in (3.4). Suppose for each $i \in \mathcal{N}$, we have $\varphi_i(x_i, y_i; x^{-i}) \equiv \varphi_i(x_i, y_i)$, i.e., assume that φ_i is independent of x^{-i} . If the infimum in (3.11) is achieved, the EPEC has an equilibrium.*

3.2 The EPEC as a shared-constraint game

The following example shows some settings where this theorem could be applied.

Example 3.2. Consider a game comprising of players that are either “firms” or “service-providers”. Service-providers take payments from contractors to complete a certain task. The payoff received by each service-provider depends on the his strategy, the strategies of all other service-providers and the payment he receives from the firm. Thus service-providers compete amongst each other in a Nash game parametrized by the payments of the firms. The payoff received by the firms depends on the resulting equilibrium amongst the service-providers and their payments to these service-providers. This payoff does not explicitly depend on the strategies of the other firms, but does so in an implicit manner through the equilibrium amongst the service-providers. Furthermore, firms are required to be consistent in their conjectures of the equilibrium amongst service-providers. The firms individually wish to optimize their payoff, but the coupling of their actions through the game of service-providers results in a noncooperative game between them.

It is evident that this is a multi-leader multi-follower game with firms as the leaders and service-providers as the followers. Suppose there are N firm, let $x_i, \varphi_i, i = 1, \dots, N$ be their strategies and payoffs respectively. Let y_i denote the tuple of strategies of the service-providers conjectured by firm i and let $\mathcal{S}(x)$ be set of Nash equilibria for the subgame played by service-providers with firms’ strategies as x . The optimization problem of firm i is as follows.

$$\begin{array}{ll}
 C_i^{cc} & \begin{array}{l} \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i) \\ \\ x_i \in X_i, \\ \text{subject to} \quad y_i \in \mathcal{S}(x), \\ y_j = y_1, \quad j = 1, \dots, N. \end{array}
 \end{array}$$

This is a shared-constraint game of the form of \mathcal{E}^{cc} . By Theorem 3.6, there is a equilibrium to this game. Note that “service-providers” in the above example could be firms that compete in a market and “firms” could be investors where x_i ’s are their investments. □

■ 3.2.4 Properties of Υ

To show the upper semicontinuity of Υ , let us recall some background from set-valued analysis. The following definitions are commonly found; see, e.g., Hogan [Hog73]. A set-valued map $S : T \rightarrow 2^W$ is *closed* at \bar{t} if $\{t_k\} \rightarrow \bar{t}$, $w_k \in S(t_k)$ and $w_k \rightarrow \bar{w}$ implies $\bar{w} \in S(\bar{t})$. It is closed on a set T if it is closed at all $\bar{t} \in T$. S is *upper semicontinuous* at \bar{t} if for any open set $U \supset S(\bar{t})$, there exists a neighbourhood V of \bar{t} such that for all $t \in V$, $S(t) \subset U$. If S is upper semicontinuous at \bar{t} it is closed at \bar{t} . If S is closed at \bar{t} and *locally bounded* at \bar{t} , i.e. if there exists a neighbourhood B of \bar{t} such that the set

$$\bigcup_{t \in B \cap T} S(t),$$

is bounded, then S is upper semicontinuous at \bar{t} . Instead of the term locally bounded, the term *uniformly compact* is often used, for obvious reasons. S is called *open* at \bar{t} if $\{t_k\} \rightarrow \bar{t}$ and $\bar{w} \in S(\bar{t})$ imply the existence of a sequence $\{w_k\}$ such that $w_k \in S(t_k)$ for k sufficiently large and $w_k \rightarrow \bar{w}$. S is open if and only if it is lower semicontinuous. S is *continuous* at \bar{t} if it is both open and closed at \bar{t} . Note that our notion of continuity is a little weaker than that in other sources, such as Aubin [AF90], which require S to be upper and lower semicontinuous for it to be continuous.

Lemma 3.7 *Let Ψ be continuous on $\mathcal{F} \times \mathcal{F}$ and assume that the infimum in (3.11) is achieved by a point $(\bar{x}, \bar{y}) \in \mathcal{F}$ for each $(x, y) \in \mathcal{F}$. Then Υ is closed and nonempty. If Υ is locally bounded, it is upper semicontinuous. If Υ is locally bounded and single-valued, then it is continuous (as a single-valued function).*

Proof : Nonemptiness of Υ follows from the assumption that the infimum is achieved. Closedness follows from classical stability results, see e.g. Hogan [Hog73], Theorem 8. The last claim is obvious as a special case of upper semicontinuity of set-valued maps for single-valued maps. ■

Clearly, local boundedness of Υ is implied by the compactness of \mathcal{F} .

In general, Υ need not be open. Indeed much simpler mappings like that the solution of a parametrized linear program can also fail to be open. Zhao, in a beautiful paper [Zha97], gave a sufficient condition for an optimization problem to have a lower semicontinuous solution set. We adapt Zhao's result to our setting to derive a condition sufficient for the lower semicontinuity of

3.2 The EPEC as a shared-constraint game

Υ . Since Υ is already known to be upper semicontinuous (if locally bounded), this condition is essentially a sufficient condition for Υ to be continuous. Let $\Psi^*(x, y) := \Psi(x, y, \Upsilon(x, y))$.

Lemma 3.8 *Suppose Ψ is continuous on $\mathcal{F} \times \mathcal{F}$ and let $(x, y) \in \mathcal{F}$ be a point. If*

1. Υ is locally bounded at (x, y) and
2. for all $\varepsilon > 0$ there exists $\alpha > 0$ and $\delta > 0$ such that

$$\forall (\hat{x}, \hat{y}) \in B((x, y), \delta), \quad \forall (\bar{x}, \bar{y}) \in \mathcal{F} \setminus B(\Upsilon(\hat{x}, \hat{y}), \varepsilon)^c, \quad \Psi(\hat{x}, \hat{y}, \bar{x}, \bar{y}) \geq \Psi^*(x, y) + \alpha,$$

then Υ is lower semicontinuous at (x, y) .

Proof : See Zhao, Theorem 1, [Zha97]. ■

The condition in 2 is necessary for the lower semicontinuity in numerous cases, particularly if Ψ is uniformly continuous. See Theorem 2 in Zhao's [Zha97] and Kien [Kie05] for sharper results.

■ 3.2.5 Comparison with the original EPEC

In the case where for every $x \in X$, $\mathcal{S}(x)$ is a singleton belonging to $\cap_{i \in \mathcal{N}} Y_i$, we get $\mathcal{F} = \{(x, y) \mid x \in X, y = \mathcal{S}^N(x)\}$ where $\mathcal{S}^N(x) = \prod_{j=1}^N \mathcal{S}(x)$. Furthermore, in the definition of Υ , one can substitute the follower equilibrium tuple in terms of the tuple of decisions of the leaders (as in Example 3.8 in Section 3.4.1). This approach is akin to the implicit programming approach for MPECs [LPR96]. i.e. for $(x, y) \in \mathcal{F}$,

$$\min_{(u, v) \in \mathcal{F}} \Psi(x, y, u, v) = \min_{u \in X} \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)).$$

Let

$$\Gamma(x) := \arg \min_{u \in X} \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)).$$

If x is a fixed point of $\Gamma : X \rightarrow 2^X$, then $(x, \mathcal{S}^N(x))$ an equilibrium of this game. Thus showing the existence of fixed points to Γ could be another approach to showing the existence of an equilibrium to EPECs with shared constraints such as \mathcal{E}^{ae} and \mathcal{E}^{cc} . For simplicity of exposition we will refer to fixed points of Γ as “equilibria” of the shared-constraint EPEC. Implicit programming approaches have also been used in for EPECs, for e.g. by Serali in [She84] and Su in [Su07]. Indeed these two

Chapter 3. Global Equilibria of Multi-leader Multi-follower Games

papers show the existence of an equilibrium to the original EPEC \mathcal{E} formed from MPECs such as L_i .

The implicit programming route also provides a way for making a comparison between an equilibrium of \mathcal{E} with those equilibria of the shared-constraint game that are obtained via the fixed point of Γ . In both [She84, Su07] \mathcal{S} is single-valued and $Y_i = \mathbb{R}^n$ for all i . Following this approach let us rewrite the original leader problem L_i in the following form.

$$\begin{array}{ll} L_i(x^{-i}, y^{-i}) & \begin{array}{ll} \text{minimize} & \varphi_i(x_i, \mathcal{S}(x); x^{-i}) \\ \text{subject to} & x_i \in X_i, \end{array} \end{array}$$

It is easy to see that an equilibrium of this game is the same as a fixed point of $\hat{\Gamma} : X \rightarrow 2^X$, where

$$\hat{\Gamma}(x) := \arg \min_{u \in X} \sum_{i=1}^N \varphi_i(u_i, \mathcal{S}(u_i, x^{-i}); x^{-i}) = \arg \min_{u \in X} \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}(u_1, x^{-1}), \dots, \mathcal{S}(u_N, x^{-N})).$$

Notice the difference between $\hat{\Gamma}$ and Γ . Importantly, observe that a fixed point of one is not necessarily a fixed point of the other. This may come as a surprise, considering that Lemmas 3.1 and 3.3 show that equilibria of the original (without shared constraints) EPEC, \mathcal{E} are also equilibria of EPECs with shared constraints of the form \mathcal{E}^{ae} and \mathcal{E}^{cc} . But this “contradiction” can be explained by noticing that fixed points of $\hat{\Gamma}$ are equilibria of the EPECs \mathcal{E}^{ae} and \mathcal{E}^{cc} , but such equilibria need not be fixed points of Γ . Since the fixed point formulation through Υ or Γ is only a sufficient condition for the existence of equilibria of \mathcal{E}^{ae} or \mathcal{E}^{cc} , there may exist equilibria of these EPEC that are not necessarily fixed points of the Γ .

However there is another interesting consequence of this comparison. The notational similarity between Γ and $\hat{\Gamma}$ suggests that under certain conditions an equilibrium of the shared-constraint EPEC obtained as a fixed point of Γ may also be an equilibrium of the original EPEC \mathcal{E} .

Theorem 3.9 *Suppose for all $x \in X$, $\mathcal{S}(x)$ is a singleton lying in $\cap_{i \in \mathcal{N}} Y_i$ and let the objectives of players be such that*

$$\Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)) \leq \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}(u_1, x^{-1}), \dots, \mathcal{S}(u_N, x^{-N})), \quad \forall u, x \in X.$$

3.3 Nonconvex fixed point theorems

Then every fixed point of Γ is also a fixed point of $\widehat{\Gamma}$ and thus an equilibrium of \mathcal{E} .

Proof : If x is a fixed point of Γ ,

$$\Psi(x, \mathcal{S}^N(x), x, \mathcal{S}^N(x)) \leq \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}^N(u)) \quad \forall u \in X.$$

By the hypothesis of the theorem, we have

$$\Psi(x, \mathcal{S}^N(x), x, \mathcal{S}^N(x)) \leq \Psi(x, \mathcal{S}^N(x), u, \mathcal{S}(u_1, x^{-1}), \dots, \mathcal{S}(u_N, x^{-N})),$$

which means x is a fixed point of $\widehat{\Gamma}$. ■

■ 3.3 Nonconvex fixed point theorems

We showed in Section 3.2 that a sufficient condition for the existence of an equilibrium to the EPEC with shared constraints, is the existence of a fixed point of a map, defined on the possibly nonconvex set \mathcal{F} . Much of fixed point theory commonly applied in mathematical programming relies on convexity and various sufficient conditions for the existence of Nash equilibria pass through the application of theorems equivalent to convex fixed point theorems of Brouwer or Kakutani. The nonconvexity of \mathcal{F} is a significant hindrance in discovering such a condition for the EPEC. This section introduces fixed point theorems that apply without the requirement of convexity of \mathcal{F} . Note that if the objective functions of players have domain that is the entire space, then additional requirements on \mathcal{F} , for which this section builds the theory, are not needed. This is discussed in Section 3.4.1.

There are many varieties of nonconvex fixed point theorems present in literature. The hypotheses of the fixed point theorems we survey do not limit the nature of the mapping. Most conditions are imposed on the nature of the domain. Central to this approach is the notion of the *fixed point property*. A set X has this property if any continuous single valued self-mapping of X admits a fixed point. So compact convex sets in Euclidean spaces have the fixed point property. Our review of nonconvex fixed point theorems is thus, essentially, a review of a class of sets, called compact absolute retracts, with the fixed point property.

Our review of nonconvex fixed point theorems sit naturally within two approaches to fixed point theory: first through algebraic topology and the second through set theoretic topology. Our exposition situates these theorems through the latter approach. One of the intentions of the authors is to bring these fixed point theorems to the attention of the community of mathematical programmers. Thus our attempt has been at keeping the survey extensive and yet relevant to mathematical programming. Throughout, our concepts are mentioned in full generality and then specialized to Euclidean spaces, and particularly constraints arising in mathematical programming, in the examples that follow.

■ 3.3.1 Fixed point property and absolute retracts

Following is the formal definition of the fixed point property.

Definition 3.1 (Fixed point property) *A topological space X is said to have the fixed point property if every continuous function that maps X to itself admits a fixed point.*

A key topological notion is that of a *retract*.

Definition 3.2 (Retract) *Let B be a topological space and $A \subseteq B$. A is said to be a retract of B if there exists a continuous function $r : B \rightarrow A$ such that $r(a) = a$ for all $a \in A$. The function r is called a retraction.*

Example 3.3. *Closed convex set in \mathbb{R}^n :* Any closed convex set, $C \subseteq \mathbb{R}^n$ is a retract of a set $B \subseteq \mathbb{R}^m$, $m \geq n$, containing it. To see this, take r in Definition 3.2 as the projection on C restricted to B . □

We introduce some nonconvex sets with this property that are not topologically too different from convex sets. Indeed if viewed in a broader topological sense these sets are seen to possess precisely those properties of convex sets that enable fixed point theorems such that Kakutani's or Brouwer's. The motivation behind our approach is as follows. The fixed point property is invariant under two topological operations: (a) under homeomorphisms – if X has the fixed point property and Y is homeomorphic to X , then Y also has the fixed point property– and (b) under retractions. If X has the fixed point property, then any retract Y of X has the fixed point property. So, to show that a set A has the fixed point property, one could show that it is homeomorphic to a retract of another

3.3 Nonconvex fixed point theorems

set B which is known to have this property. Indeed one could attempt explicitly constructing a function f such that $A = f(B)$, where f is the composition $h \circ r$ of a homeomorphism h and a retraction r . However this *modus-operandi* often proves to be as hard as the original problem.

Maps of the form $h \circ r$ are called r -maps [Bor67] and the set A above is said to be the r -image of B . A deep and systematic understanding of sets that are r -images of compact convex sets in \mathbb{R}^n is one of the consequences of Borsuk's incredible *theory of retracts*. Thanks to this, our way of identifying such sets does not need to pass through the explicit construction of an r -map and can instead be done through unions and intersection of compact convex sets. The central idea in this theory is the aforementioned notion of an absolute retract.

Let us warm up to some topological notions, beginning with the concept of *contractibility*. Recall that a homotopy H on a space X is a continuous function mapping $X \times [0, 1]$ to X . A key topological property is that of contractibility. Let $\mathbf{1}_X$ denote the identity mapping on X .

Definition 3.3 (Contractibility) *A topological space X is said to be contractible if there exists a homotopy $H : X \times [0, 1] \rightarrow X$ such that $H(\cdot, 0) = \mathbf{1}_X$ and $H(\cdot, 1) = \bar{x}$ for some $\bar{x} \in X$.*

Example 3.4.

1. *Convex set:* A convex set is contractible to any point in the set. For a convex set X and any point \bar{x} in it one may define the homotopy $H(t, \cdot) = (1 - t)\mathbf{1}_X + t\bar{x}$, where $\mathbf{1}_X$ is the identity mapping restricted to X . H maps $[0, 1] \times X$ to X because $H(t, X) \subset X$ for each $t \in [0, 1]$, by convexity of X .
2. If X is *star-shaped*, i.e. there exists a point \bar{x} , called *star-center*, in X such that the segment joining \bar{x} to any point in X lies in X , then X is contractible; it is contractible to the star-center.
3. The *complementarity feasible region* in \mathbb{R}^2 is contractible. Let $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \perp y \leq 0\}$ be this region. X is contractible to $(0, 0)$, since it is star-shaped with $(0, 0)$ as the star-center. Notice that contractibility necessitates connectedness. Hence, the complementarity feasible region with non-degeneracy or the *strict* complementarity feasible region,

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \perp y \leq 0, xy > 0\},$$

is *not* contractible. The product of contractible sets is contractible. If X and Y are contractible to x and y respectively, $X \times Y$ is contractible to (x, y) . It follows that the complementarity feasible region in \mathbb{R}^{2n} ,

$$X = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid 0 \leq x \perp y \leq 0\}$$

is contractible. □

Here we review only those notions from the theory of retracts that we need; a more complete picture is given in Borsuk [Bor67] and Hu [Hu65]. Throughout, by “space” we shall mean a topological space. We caution the reader to not confuse it with the notion of a linear or vector space. By a neighbourhood is meant an open set in the topology. A locally convex space is a topological vector space in which every point admits a convex neighbourhood. A homeomorphism between spaces X and Y is a continuous function $h : X \rightarrow Y$ such that h^{-1} exists and is continuous. In such a situation, X and Y are said to be homeomorphic. A metrizable space is a space homeomorphic to a metric space.

Definition 3.4 *1. Neighbourhood Retract:* Let X be a topological space. A set $A \subseteq X$ is said to be a neighbourhood retract of X if A is a retract of an open subset U of X .

2. Absolute Neighbourhood Retract (ANR): A metrizable space Y is said to be an ANR if every homeomorphic image of Y as a closed subset of a metrizable space Z is a neighbourhood retract of Z .

3. Absolute Retract (AR): A metrizable space Y is an AR if every homeomorphic image of Y as a closed subset of a metrizable space Z is a retract of Z .

All open subsets of X are also neighbourhood retracts of X . In keeping with this logic, one may also regard the empty set as a neighbourhood retract and since, in a topological space X the total space X is an open set, every retract of X is a neighbourhood retract. A metrizable space Y is an A(N)R if and only if for all metrizable spaces Z containing a closed subset X which is homeomorphic to Y , there exists a retraction $r : Z \rightarrow X$ ($r : U \rightarrow X$, for some U open in Z). A retract of an AR is an AR and a neighbourhood retract of an ANR is an ANR. The notion of contractibility relates ARs to ANRs.

3.3 Nonconvex fixed point theorems

Theorem 3.10 (Theorem 2.1 (IV) [DG03]) *Y is an AR if and only if it is a contractible ANR.*

Note that this implies that ARs are necessarily nonempty. We now quote the result that is central to our purpose and the role that ARs play in fixed point theory.

Theorem 3.11 (Theorem 7.4 [Hu65]) *Every compact AR has the fixed point property.*

Note that no assumption of convexity was made in the definition of ARs or in the above theorem. ARs can however be related to convex sets: every AR is the r -image of a convex subset of a normed linear space. Indeed every compact AR is the r -image of the convex hull of a finite set of points in some \mathbb{R}^n . ARs also contain familiar classes of convex sets in normed spaces.

Theorem 3.12 (Dugundji extension theorem (Theorem 7.5 (II) [DG03])) *Every metrizable convex subset of a locally convex linear space is an AR.*

Dugundji's result shows that Theorem 3.11 contains in it Brouwer's fixed point theorem. The theory of retracts provides us a way of constructing new ARs and ANRs from old ANRs and thus generalize Brouwer's fixed point theorem in a significant way. We quote below some results of this character. More such results may be found in Borsuk [Bor67, ch. IV] and the section on "pasting ANRs together" in [DG03, p. 283].

Theorem 3.13 (Theorem 6.1 p. 90 [Bor67]) *1. If X_1, X_2 are ARs and $X_1 \cap X_2 \neq \emptyset$ is an AR then $X_1 \cup X_2$ is an AR.*

2. If X_1, X_2 are ANRs and $X_1 \cap X_2$ is an ANR then $X_1 \cup X_2$ is an ANR.

Example 3.5. The complementarity feasible region in \mathbb{R}^2 , $X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \perp y \leq 0\}$ is an AR. This can be seen from the following: $X_1 = \{(x, 0) \mid x \geq 0\}$ and $X_2 = \{(0, y) \mid y \geq 0\}$ are compact convex sets in \mathbb{R}^2 . By Dugundji's extension theorem, X_1 and X_2 are ARs. Further, their intersection $X_1 \cap X_2$ is the singleton $\{(0, 0)\}$, so is an AR. By the above theorem, $X = X_1 \cup X_2$ is an AR. By the same token the bounded complementarity feasible region $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \perp y \leq 0\} \cap [0, a] \times [0, b]$ is a compact AR, and has the fixed point property. \square

ARs are less common in as constraints of EPECs than ANRs. In fact ANRs are very common and a vast variety of sets from mathematical programming can be shown to be ANRs. For our purpose, we require only the following result.

Theorem 3.14 (Aronzajn-Borsuk (Corollary 4.4, p. 283 [DG03])) *Any finite union of closed metrizable convex sets in a locally convex space in an ANR.*

Example 3.6. It follows from the above result that the solution set of a linear complementarity problem is an ANR. Thus if this solution set is contractible and compact, it has the fixed point property. \square

The following is another important property that allows one to construct new ANRs from known ANRs.

Theorem 3.15 (Proposition 1.3, p. 279 [DG03] and Theorem 7.1 [Bor67])

A finite cartesian product $\prod_{i=1}^n Y_i$ is an $A(N)R$ if and only if each Y_i is an $A(N)R$.

Example 3.7. The complementarity feasible region in \mathbb{R}^{2n} , $X = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid 0 \leq x \perp y \geq 0\}$ is the cartesian product of ARs, $\prod_{i=1}^n \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \perp y \geq 0\}$, and hence is itself an AR. \square

Relaxing compactness

We now generalize the above fixed point theorems to relax the requirement of the compactness on the domains considered. The conquest of non-compactness is non-trivial to achieve *in spirit*. For most unbounded (or open) subsets of \mathbb{R}^n , one can construct a mapping lacking a fixed point by moving the “fixed-point-to-be” towards infinity (or the boundary) [Sma74]³. In the results that follow, we will conquer non-compactness *in letter*. We allow the domain of the function to be non-compact, but require that the function have a compact image.

As an example, let C be a closed and convex in \mathbb{R}^n and $f : C \rightarrow C$ be a continuous *compact mapping*, i.e. $f(C) =: B$ is compact. Despite C not being compact, f admits a fixed point. Define $g : \text{conv } B \rightarrow \text{conv } B$ as the restriction of f to $\text{conv } B$; this is well defined, since C is convex and contains $\text{conv } B$. Brouwer’s fixed point theorem yields a fixed point to g , which by definition is also a fixed point of f . Extending this logic we get our first generalization of Theorem 3.11 to non-compact domains.

³There is a counter example to this. The set formed by the union of the graph of $x \mapsto \sin(1/x)$ over the interval $(0, 1]$ and the point $(0, 1)$, is non-compact, but has the fixed point property. See [Nad92], p. 12.

3.3 Nonconvex fixed point theorems

Theorem 3.16 *Let X be a space and $f : X \rightarrow X$ be continuous compact mapping. If $f(X)$ is an AR, f has a fixed point.*

A more sophisticated argument (see p. 6–8 [DG03]) yields another generalization of Theorem 3.11.

Theorem 3.17 (Theorem 4.6 p. 8 [DG03]) *Let X be an AR and $f : X \rightarrow X$ be a continuous compact mapping. Then f has a fixed a point.*

■ 3.3.2 Fixed point theory for nonconvex set-valued maps

We now come to the fixed point theory of set-valued maps; having prepared the background already, we can state them readily. The following theorem due to Eilenberg and Montgomery, generalizes Kakutani's fixed point theorem to nonconvex domains. ARs and the property of contractibility play an important role here too.

Theorem 3.18 (Eilenberg–Montgomery [EM46]) *Let X be a compact AR and $T : X \rightarrow 2^X$ be a set-valued map. If T is upper semicontinuous and for each $x \in X$, $T(x)$ is contractible, then T admits a fixed point.*

Analogously to single-valued maps, a set-valued map $T : X \rightarrow 2^Y$ is called compact if $T(X) = \bigcup_{x \in X} T(x)$ is contained in a compact subset of Y . A stronger version of this theorem can be articulated as follows.

Theorem 3.19 (Corollary 7.5, p. 543 [DG03]) *Let X be an AR and let T be a compact set-valued mapping of X into itself. If T is upper semicontinuous and for each $x \in X$, $T(x)$ is contractible, then T admits a fixed point.*

Another approach to establishing the existence of fixed points to set-valued mappings is by establishing the existence of “nearby” continuous functions. Let $T : X \rightarrow 2^X$ where X is a normed space. A continuous single-valued function f is called an ϵ -approximation of T if the graph of f is contained in an ϵ neighbourhood of the graph of T . i.e.

$$\forall x \in X, \quad \exists x' \in X, y' \in T(x') \quad \text{s.t.} \|(x, f(x)) - (x', y')\| < \epsilon. \quad (3.12)$$

f is also called a *graphical approximation* of T . If X is a compact AR the ϵ -approximation f admits a fixed point. Using tighter ϵ -approximations, one may conclude the existence of a fixed point to T .

Theorem 3.20 (p. 108, Theorem 22.4, [Gór99]) *Let X be a metric space $T : X \rightarrow 2^X$ be a multi-valued upper semicontinuous mapping. If for each $\epsilon > 0$ there exists an ϵ -approximation of T , T admits a fixed point.*

■ 3.4 Fixed point theory for Υ

In the remainder of this chapter we deal with only shared-constraint games. We use “ Ω ” to denote a shared constraint. By this is meant that Ω is either Ω^{ae} or Ω^{cc} or the game is of the form of \mathcal{G}^{ind} . We use \mathcal{F} to denote the fixed point set of Ω . All objective functions are defined to be continuous functions $\varphi_i : \mathcal{F} \rightarrow \mathbb{R}, i \in \mathcal{N}$. For any function $f : \mathcal{F} \rightarrow \mathbb{R}$, we say it is *convex* if the following is true

1. For $z_1, z_2 \in \mathcal{F}$ such that the segment joining z_1, z_2 belongs to \mathcal{F} ,

$$f(\alpha z_1 + (1 - \alpha)z_2) \leq \alpha f(z_1) + (1 - \alpha)f(z_2) \quad \forall \alpha \in (0, 1)$$

2. For any $z_1, z_2 \in \mathcal{F}$,

$$f(z_1) \geq f(z_2) + \nabla f(z_2)^T(z_1 - z_2)$$

We say it is strictly convex if the inequalities above hold strictly.

In this section, we apply fixed point theorems from Section 3.3 to Υ . These theorems are broad and may be directly applied to Υ and based on this, sufficient conditions for existence of equilibria may be stated. There are two reasons for the necessity of this section. First, the reach of such directly obtained sufficient conditions for a map like Υ , with a specific structure, is not immediate. Theorems in Section 3.3 are stated in terms of abstract concepts, and these concepts need to be interpreted for Υ . The second reason is to understand the barriers to generalization of these results and thereby identify pathology, which if weeded out, could yield more general results.

3.4 Fixed point theory for Υ

Throughout we assume that \mathcal{F} is compact. This implies that the infimum in definition of Υ , (3.11), is achieved and Υ is upper semicontinuous. The case of unbounded \mathcal{F} can be dealt with by assuming Υ to be a compact mapping and applying the rest of the results below *in toto*. We also assume that \mathcal{F} is an absolute retract; sufficient conditions for this are postponed to Section 3.5, where specific models will be considered. It will be convenient at times to take \mathcal{S} to be the solution set of a linear complementarity problem.

$$\mathcal{S}(x) = \text{SOL}(\text{LCP}(M, Px + q)). \quad (3.13)$$

■ 3.4.1 Broad results

The main fixed point theorems of Section 3.3 are those of Borsuk (Theorem 3.11), which states that continuous single-valued self-mappings of absolute retracts have fixed points, and of Eilenberg and Montgomery (Theorem 3.18), which states that upper semicontinuous mappings on compact absolute retracts with contractible values admit fixed points. The following theorem is a direct consequence of Theorems 3.11 and 3.18.

Theorem 3.21 *Let \mathcal{F} be compact and suppose that Υ is either*

1. *single-valued on \mathcal{F} , or*
2. *multi-valued on \mathcal{F} with contractible values.*

Then if \mathcal{F} is a absolute retract, \mathcal{E} has an equilibrium.

Assuming that \mathcal{F} is an absolute retract, the reach of this result is determined by the the values mapped by Υ . Observe that \mathcal{F} is the feasible region of an MPEC and $\Upsilon(x, y)$ is the set of global minimizers of an MPEC parametrized by (x, y) . Thus assessing the values mapped by Υ amounts to examining the global minimizers of an MPEC. We now examine each of these requirements and identify open questions. Folloing that we examine the requirement of \mathcal{F} being an absolute retract.

Single-valuedness of Υ

Asking that Υ be single-valued on \mathcal{F} is asking that this MPEC in (3.11) have a unique global minimizer for all values of the parameter (x, y) belonging to \mathcal{F} . We do not know of any general

Chapter 3. Global Equilibria of Multi-leader Multi-follower Games

result in literature that guarantees this. However a trivial sufficient condition can be derived.

Lemma 3.22 *Let Z be a set in \mathbb{R}^n and let $f : Z \rightarrow \mathbb{R}$. Assume that f is a continuously differentiable and strictly convex function and let $z^* \in Z$ be a global minimizer of f over Z . Then z^* is the only global minimizer of f over Z if it solves the variational inequality:*

$$\nabla f(z^*)^T(z - z^*) \geq 0 \quad \forall z \in Z.$$

Proof : Suppose there are two global minimizers z^* and x^* in Z . Since f is strictly convex

$$f(x^*) > f(z^*) + \nabla f(z^*)(x^* - z^*).$$

But since z^* solves the variational inequality, we get $f(x^*) > f(z^*)$, a contradiction. ■

Observe that every solution of the variational inequality above is a global minimizer; strict convexity of the objective provides that this solution is unique. For Υ to be single-valued, this would mean that $\Psi(x, y, \cdot)$ be strictly convex for all $(x, y) \in \mathcal{F}$ and the parameterized variational inequality, over a possibly nonconvex set \mathcal{F} has a solution.

However MPECs may have unique solutions. Indeed, Sherali et al. [SSM83] show that for a Stackelberg-Nash-Cournot game with a single leader the leader's MPEC reduces to a convex optimization problem. This relies on getting an expression for the follower equilibrium, which is not available except for specific models. We reproduce this example here.

Example 3.8. This game has one Stackelberg leader and n identical followers. Leader solves problem (L) and each follower solves problem $(F(y, x))$, where y denotes the equilibrium strategy of any follower, whereby at the Nash equilibrium of followers we have $y \in \text{SOL}(F(y, x))$. Below, a, b, c, d are positive real numbers.

(L)		minimize	$\frac{1}{2}dx^2 - x(a - b(x + ny))$		
		subject to	$y \in \text{SOL}(F(y, x)),$		
			$x \geq 0.$		

3.4 Fixed point theory for Υ

$$\begin{aligned} (F(y, x)) \quad & \underset{\bar{y}}{\text{minimize}} \quad \frac{1}{2}c\bar{y}^2 - \bar{y}[a - b(\bar{y} + x + (n-1)y)] \\ & \text{subject to} \quad \bar{y} \geq 0. \end{aligned}$$

For any x , there is a unique y that satisfies $y \in \text{SOL}(F(y, x))$, given by

$$y = \begin{cases} (a - bx)/(c + b(n+1)) & \text{if } 0 \leq x \leq a/b, \\ 0 & x > a/b. \end{cases}$$

The uniqueness of the solution of the MPEC (L) follows from an argument [SSM83] that upon substitution of this y in (L), (L) reduces to the following convex program.

$$\begin{aligned} (C) \quad & \underset{x}{\text{minimize}} \quad \frac{1}{2}dx^2 - x \left[a - b \left(x + n \frac{(a-bx)}{(c+b(n+1))} \right) \right] \\ & \text{subject to} \quad 0 \leq x \leq a/b. \end{aligned}$$

This program has a unique optimal solution, $x = \frac{a(b+c)}{2b(b+c) + d(b+c) + bdn}$. □

More intuition can be given when \mathcal{S} is the map in (3.13). It is easy to show (see Section 3.5) that \mathcal{F} is a union of finitely many convex sets, determined by the various possible active sets of the LCP. Indeed one may write \mathcal{F} as a union of finitely many active sets $\bigcup_{A \in \mathcal{A}} \mathcal{F}_A$, where \mathcal{A} is the collection of all possible active sets. Furthermore, for $(x, y) \in \mathcal{F}$, $\Upsilon(x, y)$ is given by the minimum of the minimizers over each active set

$$\Upsilon(x, y) = \left\{ (\bar{x}, \bar{y}) \in \mathcal{F} \mid \Psi(x, y, \bar{x}, \bar{y}) = \min_{A \in \mathcal{A}} \min_{(u, v) \in \mathcal{F}_A} \Psi(x, y, u, v) \right\} = \bigcup_{A \in \mathcal{A}} \Upsilon_A(x, y),$$

where $\Upsilon_A(x, y)$ consists of points from $\Upsilon(x, y)$ that are feasible with respect to active set \mathcal{F}_A . Therefore if $\Psi(x, y, \cdot)$ is convex, $\Upsilon_A(x, y)$ is a convex (if nonempty) set for each $A \in \mathcal{A}$ and $\Upsilon(x, y)$ is a union of convex sets. If $\Psi(x, y, \cdot)$ is strictly convex, $\Upsilon(x, y)$ is a union of finitely many points, since $\Upsilon_A(x, y)$ is at most a singleton. $\Upsilon(x, y)$ is contained in the set of minimizers of $\Psi(x, y, \cdot)$ over each of these convex sets. For $\Upsilon(x, y)$ to be a singleton, the minimizer that leads to the least value of $\Psi(x, y, \cdot)$ should be unique.

Contractibility of $\Upsilon(x, y)$

When \mathcal{S} is given by (3.13) and $\Psi(x, y, \cdot)$ is convex, $\Upsilon(x, y)$ is a union of convex (if nonempty) sets $\Upsilon_A(x, y)$. If the union $\Upsilon(x, y)$ is a convex set, it is contractible. If these convex sets share a common point, then $\Upsilon(x, y)$ is star-shaped and thus contractible. A classical theorem of Helly (see Rockafellar, Ch. 21 [Roc97]) states that these convex sets share a common point if every $n + 1$ of them does, where n is the dimension of the space of \mathcal{F} .

Note that Υ_A is also upper semicontinuous, if it is nonempty. Therefore if there exists an active set $A \in \mathcal{A}$ such that $\Upsilon_A(x, y)$ is nonempty for all $(x, y) \in \mathcal{F}$, then Υ_A admits a fixed point. Thus Υ admits a fixed point. Furthermore, one does not need \mathcal{F} to be an absolute retract in this case.

Theorem 3.23 *Let $\Psi(x, y, \cdot)$ be convex for each $(x, y) \in \mathcal{F}$ and let \mathcal{S} be given by (3.13). If \mathcal{F} is compact and there exists an active set A such that $\Upsilon_A(x, y)$ is nonempty for all $(x, y) \in \mathcal{F}$, then there exists an equilibrium to the EPEC.*

Proof : Let A be the active set mentioned in the claim. Consider the map $\Upsilon_{A|\mathcal{F}_A}$ which is the restriction of Υ_A to \mathcal{F}_A . $\Upsilon_{A|\mathcal{F}_A}$ is upper semicontinuous with convex compact domain (\mathcal{F}_A) and convex compact values. Kakutani's fixed point theorem, yields a fixed point to $\Upsilon_{A|\mathcal{F}_A}$. This fixed point is also a fixed point of Υ_A , and hence Υ . ■

Extension of φ_i 's

We now come to an important technical point. We assume for ease of exposition, that $X \times Y$ which is convex, is also compact. Recall the definition of Υ as a map from \mathcal{F} to subsets of \mathcal{F} . The domain of Υ was taken to be \mathcal{F} , which was not necessarily convex, and this implied the need for advanced fixed point theory. Furthermore, in our game φ_i 's had domain \mathcal{F} . We wish to point out here that (a) when the φ_i 's have domain \mathcal{F} , then \mathcal{F} being an absolute retract is needed not only for topological fixed point theory but also to keep the modified reaction map Υ well defined and (b) the use of advanced fixed point theory and particularly the introduction of absolute retracts can be circumvented if the φ_i 's are assumed to be defined over a larger space.

Let us consider (a) first. In a game, it is customary (and well posed) to define the objective functions with their domain as the strategy space. For a shared-constraint game, the strategy

3.4 Fixed point theory for Υ

space is the set defined by this constraint (see Başar et al. [BO99]). Our objective functions φ_i are also chosen in this manner as maps from \mathcal{F} to \mathbb{R} . Note however that Ψ , as defined in (3.10), may not be well-defined since there may be points (x, y) and (\bar{x}, \bar{y}) in \mathcal{F} such that for some i , $(\bar{x}_i, \bar{y}_i, x^{-i}, y^{-i}) \notin \mathcal{F}$, whereby the term ' $\varphi_i(\bar{x}_i, \bar{y}_i; x^{-i})$ ' in the definition of Ψ is not defined. But if \mathcal{F} is an AR, then every continuous function $f : \mathcal{F} \rightarrow \mathbb{R}$ can be extended to a continuous $\tilde{f} : \mathbb{R}^m \rightarrow \mathbb{R}$, where \mathcal{F} is a closed subset of \mathbb{R}^m , such \tilde{f} when restricted to \mathcal{F} is equal to f ; see Borsuk, [Bor67], p. 87. Therefore we may assume without loss of generality that φ_i 's are defined over the entire space of \mathbb{R}^m and thus our analysis holds.

Now consider (b). While posing the game, if one considers only those objective functions that are well defined over a larger set G , where G is large enough for Ψ to be well defined on G then Υ could also be defined over G . In particular, if φ_i 's have domain $G = X \times Y$, Ψ is well defined over G and Υ could be redefined as a map $\tilde{\Upsilon}$ from a compact convex set, i.e. $X \times Y$, to subsets of \mathcal{F} , which are subsets of $X \times Y$:

$$\tilde{\Upsilon}(x, y) = \{(\bar{x}, \bar{y}) \mid \Psi(x, y, \bar{x}, \bar{y}) = \inf_{(u, v) \in \mathcal{F}} \Psi(x, y, u, v)\} \quad (x, y) \in X \times Y.$$

Observe that this redefinition does not alter fixed points of Υ ; i.e. fixed points of $\tilde{\Upsilon}$ are the same as fixed points of Υ and are thus equilibria of the EPEC. This redefinition puts $\tilde{\Upsilon}$ within the applicability of conventional “convex” fixed point theorems such as Brouwer’s or Kakutani’s and the advanced theorem of Eilenberg and Montgomery, *without* placing additional assumptions on the shape of \mathcal{F} . To apply any of these theorems, the nature of images of $\tilde{\Upsilon}$ need to be of the kind required by the theorems. So Brouwer’s fixed point theorem would require single-valuedness of $\tilde{\Upsilon}$, Kakutani’s would require compact convex valuedness and Eilenberg and Montgomery’s theorem would require contractible valuedness.

It must be noted that the liberty to define φ_i 's over a larger space doesn't always exist. When one does not have that liberty, absolute retracts provide a way for formal extension of the φ_i 's over a larger space. Note also that this issue does not arise explicitly when the strategy space is closed and convex, since closed convex sets *are* absolute retracts. Consequently we may say that when an extension of φ_i 's cannot be assumed to exist, absolute retracts provide a way to fixed point theory as well as an extension of φ_i 's.

■ 3.4.2 Refined results and pathology

Theorem 3.21 was derived from a direct application of fixed point theorems to Υ . But since Υ has a more specific structure, there may be a possibility of obtaining more refined results. We have explored this possibility while staying within the framework of Theorem 3.21. Here we will move beyond Theorem 3.21. Assuming that Υ is neither single-valued nor contractible valued, there are two other broad principles on which the existence of its fixed points may rest. These two closely related principles are based on using related single-valued maps to claim existence of fixed points to set-valued maps. The first idea is that of approximation, mentioned in (3.12) on which Theorem 3.20 rests. Another closely related idea is that of a *selection*. Let $T : X \rightarrow 2^X$ be a set-valued map with nonempty values. A function $f : X \rightarrow X$ is called a selection of T if $f(x) \in T(x)$ for all $x \in X$. If X has the fixed point property and T admits a continuous selection, T admits a fixed point.

The selection approach will provide an avenue towards deriving an existence result, but will impose further requirements to make Υ continuous. In the approximation approach, we will approximate Υ by continuous single-valued maps.

Selection of Υ

The conventional approach to showing the existence of selection to a set-valued map is by construction. A distinguished point is chosen from each mapped value of the set-valued map based on a criterion. If the criterion is so chosen, the resulting function of distinguished points turns out to be continuous. A common criterion is the minimal selection in which one picks the element of least norm. When the set-valued map takes closed convex values, this indeed results in a unique element. In the nonconvex case we need to make an assumption of “nondegeneracy” of the following kind.

Assumption 3.1 (Nondegeneracy) *Let $\mathcal{F} \subseteq \mathbb{R}^n$. We say Υ is nondegenerate if there exists $(x^{\text{ref}}, y^{\text{ref}}) \in \mathbb{R}^n$ such that the problem $SEL(x, y)$ below has a unique minimizer for each $(x, y) \in \mathcal{F}$.*

$SEL(x, y)$	$\begin{aligned} &\underset{\bar{x}, \bar{y}}{\text{minimize}} && \ (\bar{x}, \bar{y}) - (x^{\text{ref}}, y^{\text{ref}})\ \\ &\text{subject to} && (\bar{x}, \bar{y}) \in \Upsilon(x, y). \end{aligned}$
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3.4 Fixed point theory for Υ

For the following result we require that Υ is continuous. A sufficient condition for the continuity of Υ was given in Lemma 3.8.

Theorem 3.24 *Let \mathcal{F} be compact and Υ be continuous on \mathcal{F} . Then if Υ is degenerate, the mapping*

$$(x, y) \mapsto \text{SOL}(\text{SEL}(x, y)),$$

is single-valued and a continuous selection of Υ . If \mathcal{F} is an absolute retract, the game admits an equilibrium.

Proof : It follows from Hogan, Corollary 8.1, [Hog73], that $\text{SOL}(\text{SEL}(x, y))$ is a single-valued continuous function of (x, y) . Furthermore, $\text{SOL}(\text{SEL}(x, y)) \in \Upsilon(x, y)$ for each (x, y) , whereby it is a continuous selection of Υ . When \mathcal{F} is a compact absolute retract, there exists a fixed point $\mathcal{F} \ni (\hat{x}, \hat{y}) = \text{SOL}(\text{SEL}(\hat{x}, \hat{y}))$, implying that (\hat{x}, \hat{y}) is a fixed point of Υ . ■

Remark : The nondegeneracy assumption demands that we can find a point $(x^{\text{ref}}, y^{\text{ref}})$ such that for each (x, y) no two points from $\Upsilon(x, y)$ are equi-distant from it. When \mathcal{S} is given by (3.13) and $\Psi(x, y, \cdot)$ is strictly convex for each $(x, y) \in \mathcal{F}$, $\Upsilon(x, y)$ is a finite set for (x, y) , and this requirement does not seem hard to fulfill. One may ask for a advanced version of nondegeneracy too. We may take $(x^{\text{ref}}, y^{\text{ref}})$ be a continuous function of $(x, y) \in \mathcal{F}$, rather than a constant. □

Approximation of Υ

We return to the case of upper semicontinuous Υ and comment on a pathology that prevents such an Υ from admitting a continuous selection. Assume \mathcal{F} is a compact absolute retract. The upper semicontinuous mapping Υ has the suprising property that it admits a selection $f : \mathcal{F} \rightarrow \mathcal{F}$ of the following kind. There exists a sequence of continuous functions $f_n : \mathcal{F} \rightarrow \mathcal{F}$ such that $f_n \rightarrow f$ pointwise as $n \rightarrow \infty$ (see [Sri93]). f is not necessarily continuous. But recall that if $f_n \rightarrow f$ *uniformly* then f would be continuous. For large enough n , these f_n would amount to graphical approximations of Υ in the sense of (3.12).

Furthermore, since \mathcal{F} is an absolute retract, there exists for each n , $z_n \in \mathcal{F}$ such that $z_n = f_n(z_n)$. Without loss of generality we may assume $\{z_n\}$ to be convergent to z and confined to a subset B of \mathcal{F} . If $f_n \rightarrow f$ uniformly on B , it follows that $f(z) = z$. i.e. f , and hence Υ has a fixed point.

One may ask: what is a sufficient condition for these f_n to converge uniformly to f ? Such a condition is hard to provide. However probabilistic answer can be given. Let μ be any probability measure on \mathcal{F} . The classical theorem of Egorov states for all $\epsilon > 0$, there exists $E \subseteq \mathcal{F}$ measurable such that $\mu(E) > 1 - \epsilon$ and $f_n \rightarrow f$ uniformly on E .

■ 3.5 Contractibility of \mathcal{F}

Section 3.4 applied the fixed point theory from Section 3.3 to Υ assuming that \mathcal{F} is an absolute retract. This section asks following question: what sort of sets \mathcal{F} are absolute retracts? As in Section 3.4, we will consider the case where \mathcal{S} is the solution map of a parametrized LCP, as in (3.13). In this case \mathcal{F} is a union of convex sets. We derive a sufficient condition for \mathcal{F} to be star-shaped, which implies contractibility.

■ 3.5.1 EPECs arising from competing bilevel problems

Let us consider, to begin with, the sets from the game of leaders solving bilevel optimization problems. Consider the EPEC in which player i solves the following bilevel optimization problem.

$$\begin{array}{ll}
 L_i^{bl}(x^{-i}, y^{-i}) & \begin{array}{l} \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}, y^{-i}) \\ \\ \text{subject to} \quad x_i \in X_i, \\ y_i \in Y_i, \\ y_i \in \widehat{\mathcal{S}}_i(x_i). \end{array}
 \end{array}$$

Let $\widehat{\mathcal{S}}_i(x_i)$ be the solution set of the LCP below

$$\widehat{\mathcal{S}}_i(x_i) = \text{SOL}(\text{LCP}(M_i, P_i x_i + q_i)).$$

Therefore,

$$\mathcal{F}^{bl} = \prod_{i=1}^N \mathcal{F}_i^{bl}, \quad \mathcal{F}_i^{bl} = \{(x_i, y_i) \mid x_i \in X_i, y_i \in Y_i, \quad 0 \leq y_i \perp M_i y_i + P_i x_i + q_i \geq 0\}. \quad (3.14)$$

3.5 Contractibility of \mathcal{F}

\mathcal{F}_i^{bl} has a familiar structure. It is a union of closed convex sets for each i . To show this, we will momentarily introduce some notation. Suppose $y_i \in \mathbb{R}^{n_i}$ and let \mathcal{P} denote the set of all subsets of $\{1, \dots, n_i\}$. For each $\beta \in \mathcal{P}$ let $\bar{\beta}$ denote the set $\mathcal{P} \setminus \beta$ and for a vector $v \in \mathbb{R}^{n_i}$ let v^β denote the subvector of v with components that belong to β . Using this notation we see that \mathcal{F}_i^{bl} is expressible as the following finite union (this can be proved by showing each side of the equation is a subset of the other).

$$\mathcal{F}_i^{bl} = \bigcup_{\beta \in \mathcal{P}} \left\{ (x_i, y_i) \mid x_i \in X_i, y_i \in Y_i, y_i^\beta = 0, (M_i y_i + P_i x_i + q_i)^\beta \geq 0, \right. \\ \left. y_i^{\bar{\beta}} \geq 0, (M_i y_i + P_i x_i + q_i)^{\bar{\beta}} = 0 \right\}.$$

Each set in this union is a convex set. It follows then from Theorem 3.14 that \mathcal{F}_i^{bl} is an ANR for each i and from Theorem 3.15 that \mathcal{F}^{bl} is an ANR.

Lemma 3.25 *The set \mathcal{F}^{bl} given by (3.14) is an ANR.*

Thus if each \mathcal{F}_i^{bl} is also contractible, \mathcal{F} is an AR, cf. Theorem 3.10. We show in the following lemma that \mathcal{F}_i^{bl} is star-shaped if a certain condition holds, from which its contractibility follows.

Lemma 3.26 *Let \mathcal{F}^{bl} be given by (3.14), $0 \in \prod Y_i$, and suppose there exist, for each i , points $\bar{x}_i \in X_i$ such that $P_i \bar{x}_i + q_i = 0$. Then \mathcal{F}^{bl} is contractible and an AR.*

Proof : We show that for each i , \mathcal{F}_i^{bl} is star-shaped with $x_i = \bar{x}_i$ and $y_i = 0$ as the star center. Let $(x_i, y_i) \in \mathcal{F}_i^{bl}$ and $t \in [0, 1]$. Since X_i is convex, the point $tx_i + (1-t)\bar{x}_i$ belongs to X_i , and by the convexity of Y_i , ty_i lies in Y_i . We show that the homotopy H

$$H(t, (x_i, y_i)) = (tx_i + (1-t)\bar{x}_i, ty_i),$$

contracts \mathcal{F}_i^{bl} to $(\bar{x}_i, 0)$. We have $0 \leq y_i \perp M_i y_i + P_i x_i + q_i \geq 0$, and thus

$$0 \leq ty_i \perp M_i(ty_i) + P_i(tx_i) + tq_i \geq 0.$$

Since $P_i \bar{x}_i + q_i = 0$, the above relation is equivalent to

$$0 \leq ty_i \perp M_i(ty_i) + P_i(tx_i) + tq_i + (1-t)[P_i \bar{x}_i + q_i] \geq 0.$$

But this means that the point $(ty_i, tx_i + (1-t)\bar{x}_i)$ lies in \mathcal{F}_i^{bl} , because the above equation is the same as

$$0 \leq ty_i \perp M_i(ty_i) + P_i[tx_i + (1-t)\bar{x}_i] + q_i \geq 0.$$

Thus \mathcal{F}_i^{bl} is star-shaped. The product \mathcal{F}^{bl} is also star-shaped and hence contractible. We have already seen that \mathcal{F}^{bl} is an ANR, so its contractibility implies that it is an AR. \blacksquare

We have the following existence result that combines Theorem 3.21 and Lemma 3.26. In the result below we understand Υ as being defined with $\mathcal{F} = \mathcal{F}^{bl}$.

Theorem 3.27 *Consider a game in which each leader i solves a bi-level problem L_i^{bl} with constraint \mathcal{F}_i^{bl} . Suppose that for each leader i , $0 \in Y_i$ and there exist $\bar{x}_i \in X_i$ such that $P_i \bar{x}_i + q_i = 0$ and assume that either \mathcal{F}^{bl} is compact or Υ is a compact mapping. Then, the game has an equilibrium if either of the following conditions hold:*

1. Υ is single-valued on \mathcal{F}^{bl}
2. Υ is contractible-valued on \mathcal{F}^{bl} .

■ 3.5.2 EPECs with repeated equilibrium constraints

We now come to EPECs formed from leader optimization problems of the kind where all equilibrium constraints are present in each leader's problem.

$ \begin{aligned} L_i^{ae}(x^{-i}, y^{-i}) \quad & \underset{x_i, y_i}{\text{minimize}} \quad \varphi_i(x_i, y_i; x^{-i}) \\ & x_i \in X_i \\ \text{subject to} \quad & y_i \in Y_i \\ & 0 \leq y_j \perp My_j + Px + q \geq 0, \quad j = 1, \dots, N. \end{aligned} $

3.5 Contractibility of \mathcal{F}

The feasible region of this EPEC is

$$\mathcal{F} = \{(x, y) \in X \times Y \mid y \in \mathcal{S}^N(x)\}. \quad (3.15)$$

where $\mathcal{S}(x) = \text{SOL}(\text{LCP}(M, Px + q))$ and $\mathcal{S}^N(x) = \prod_{j=1}^N \mathcal{S}(x)$. Here too \mathcal{F} is the union of finitely many convex sets. To see this, observe that $\mathcal{S}^N(x)$ is the solution set of the LCP($\mathbf{M}, \mathbf{P}x + \mathbf{q}$), where $\mathbf{M} \in \mathbb{R}^{rN \times rN}$, $\mathbf{P} \in \mathbb{R}^{rN \times m}$, $\mathbf{q} \in \mathbb{R}^{rN}$ are as follows ($r = \dim(y_i)$ and $m = \dim(x)$).

$$\mathbf{M} = \begin{bmatrix} M & 0 & \dots & 0 \\ 0 & M & 0 & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & M \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} P \\ P \\ \vdots \\ P \end{bmatrix} \quad \mathbf{q} = \begin{bmatrix} q \\ q \\ \vdots \\ q \end{bmatrix}$$

Then by arguing as in the case of bilevel programs, \mathcal{F} is a union of convex sets and thus an ANR. The same argument as in Lemma 3.26, we get a condition for the contractibility of \mathcal{F} .

Lemma 3.28 *Let \mathcal{F} be given by (3.15), $0 \in Y$, and suppose there exists a tuple of leader strategies $\bar{x} \in X$ such that $P\bar{x} + q = 0$. Then \mathcal{F} is contractible and an AR.*

Proof : It is easy to see that $(0, \bar{x}) \in \mathcal{F}$. The homotopy H given by

$$H(t, (x, y)) = (tx + (1 - t)\bar{x}, ty),$$

satisfies $H(t, \mathcal{F}) \subseteq \mathcal{F}$ for each $t \in [0, 1]$. The proof for this follows exactly as in the proof of Lemma 3.26. Since $H(1, \cdot) = \mathbf{1}_{\mathcal{F}}$ and $H(0, \cdot) = (\bar{x}, 0)$, \mathcal{F} is contractible. ■

And as a consequence we have the following existence result.

Theorem 3.29 *Consider a game in which each leader i solves the problem L_i^{ae} above. Suppose that $0 \in Y$ and there exists a tuple of leader strategies $\bar{x} \in X$ such that $P\bar{x} + q = 0$ and assume that either \mathcal{F} is compact or Υ is a compact mapping. Then, the game has an equilibrium if either of the following conditions hold:*

1. Υ is single-valued on \mathcal{F}

2. Υ is contractible-valued on \mathcal{F}

Observe that \mathcal{F} is also the feasible for the EPEC formed by leaders with objectives independent of the follower decisions.

$$\begin{array}{ll}
 L_i^{ind}(x^{-i}, y^{-i}) & \text{minimize}_{x_i, y_i} \quad \varphi_i(x_i; x^{-i}) \\
 & x_i \in X_i \\
 & \text{subject to} \quad y_i \in Y_i \\
 & 0 \leq y_i \perp My_i + Px + q \geq 0.
 \end{array}$$

So Theorem 3.29 applies to this game too.

Theorem 3.30 Consider a game \mathcal{E}^{ind} where each player solves problem L^{ind} . If \mathcal{F} and Υ satisfy the conditions in Theorem 3.29, the game \mathcal{E}^{ind} has an equilibrium.

■ 3.5.3 EPEC with consistent conjectures

$$\begin{array}{ll}
 L_i^{cc}(x^{-i}, y^{-i}) & \text{minimize}_{x_i, y_i} \quad \varphi_i(x_i, y_i; x^{-i}) \\
 & x_i \in X_i \\
 & y_i \in Y_i \\
 & \text{subject to} \quad 0 \leq y_i \perp My_i + Px + q \geq 0 \\
 & y_j = y_1, \quad j = 1, \dots, N.
 \end{array}$$

The feasible region of this EPEC is

$$\mathcal{F}^{cc} = \{(x, y) \in X \times Y \mid y \in \text{SOL}(\text{LCP}(\mathbf{M}, \mathbf{P}x + \mathbf{q})) \cap \mathbf{A}\}, \quad (3.16)$$

where \mathbf{A} is the set defined in (3.8). \mathcal{F}^{cc} is also a union of convex sets since it can be rewritten as $\mathcal{F} \cap \{\mathbb{R}^m \times \mathbf{A}\}$, where \mathcal{F} is as in (3.15).

Lemma 3.31 Let \mathcal{F}^{cc} be given by (3.16), $0 \in Y$, and suppose there exists a tuple of leader strategies $\bar{x} \in X$ such that $P\bar{x} + q = 0$. Then \mathcal{F}^{cc} is contractible and an AR.

3.6 Conclusions

Proof : Once again we show that \mathcal{F}^{cc} is star-shaped with $(\bar{x}, 0)$ as the star center. First observe that $(\bar{x}, 0) \in \mathcal{F}^{cc}$, since $0 \in \mathbf{A}$. The homotopy H given by

$$H(t, (x, y)) = (tx + (1 - t)\bar{x}, ty),$$

contracts \mathcal{F} to $(\bar{x}, 0)$. Furthermore, since $t\mathbf{A} \subseteq \mathbf{A}$ for each $t \in [0, 1]$, this homotopy also contracts $\{\mathbb{R}^m \times \mathbf{A}\}$ to $(\bar{x}, 0)$. Therefore H satisfies $H(t, \mathcal{F}^{cc}) \subseteq \mathcal{F}^{cc}$ for each $t \in [0, 1]$. It follows that \mathcal{F}^{cc} is contractible. ■

We thus have an existence result for EPECs with consistent conjectures.

Theorem 3.32 *Consider a game in which each leader i solves the problem L_i^{cc} above. Suppose that $0 \in Y$ and there exists a tuple of leader strategies $\bar{x} \in X$ such that $P\bar{x} + q = 0$ and assume that either \mathcal{F}^{bl} is compact or Υ is a compact mapping. Then, the game has an equilibrium if either of the following conditions hold:*

1. Υ is single-valued on \mathcal{F} ,
2. Υ is contractible-valued on \mathcal{F} .

■ 3.6 Conclusions

In this chapter, we considered a multi-leader multi-follower game and examine the question of when a global equilibrium exists. A standard approach through the reaction map has many hindrances because of the lack of continuity in the solution set of the equilibrium constraint. We observe that these challenges are alleviated partially in EPECs with shared constraints and present a modified formulations that result in such EPECs. In this setting, a sufficient condition for the existence of a solution to such an EPEC was shown to be the existence of a fixed point to a certain modified reaction map. Sufficient conditions for this map to admit fixed points were given based on topological fixed point theory. The techniques were applied to a set of LCP-constrained multi-leader multi-follower games. Here, sufficient conditions for the contractibility of its domain were given for this class of EPECs using the theory of retracts. Finally, some open questions emerging out of our exploration were identified.

Chapter 4

Generalized Nash Games and Variational Inequalities

■ 4.1 Introduction

We have seen that variational inequalities (VIs) provide an avenue for compactly articulating equilibrium conditions for continuous-strategy Nash games. When these games are generalized to allow for coupled strategy sets, the variational conditions of the resulting *generalized* Nash games, are given by a quasi-variational inequality (QVI). In [Har91], Harker examined this class of games and showed that if a set satisfying certain conditions exists, a solution to a VI defined over this set provides a solution to the QVI. Yet, as observed by Facchinei et al. [FFP07], this result is difficult to apply to QVIs that arise generalized Nash games with shared constraints. The root of this difficulty is that even for the simplest of shared constraints, a set satisfying Harker's conditions is extraordinarily hard to find.

Motivated by these observations, this work revisits Harker's conditions in depth to qualify their applicability to shared-constraint games. Specifically, when Harker's result is applied to QVIs arising from shared-constraint games, we show that there is at most one set that satisfies Harker's conditions and has a nonempty interior: it is the set that defines the shared constraint. Furthermore, if this set satisfies Harker's conditions, then it has to be cartesian in nature; in other words, Harker's result cannot be directly applied to most nontrivial shared-constraint games¹. As a result, we are

¹Nontrivial shared-constraint games allude to games where the set defining the shared constraint is not cartesian or rectangular.

4.1 Introduction

able to formally explain why it remains a challenge to find a set satisfying Harker's conditions.

While it appears that Harker's result has reduced utility in the context of shared-constraint games, we observe that its applicability to nontrivial shared-constraint games can be salvaged through an application to a 'modified' QVI. Specifically, we construct a modified QVI that admits the same solution set as the original QVI, and for this modified QVI, there always exists a set satisfying Harker's conditions. In short, Harker's result applies to all nontrivial shared constraint games when it is applied through the modified QVI, though it fails when applied directly. As a corollary we also see that Harker's result *implies* a result provided by Facchinei et al. [FFP07] as an alternative to Harker's result.

While QVIs are natural tools for modeling game-theoretic problems, these objects are analytically harder to handle than VIs. Therefore, any results that reduce the analysis to a VI are particularly important and those provided by Harker and Facchinei et al. are, to the best of our knowledge, the only ones that enable such a reduction. We believe that our work clarifies their reach, revealing a surprising relationship between these two results, and allowing for the application of Harker's result through the modified QVI.

We end this section with a formal description of the game. The canonical game we consider is the same as the game \mathcal{G} from Chapter 2 formed from a set of players $\mathcal{N} = \{1, 2, \dots, N\}$, where player i solves the problem

$$\begin{array}{ll} A_i(x^{-i}) & \underset{x_i}{\text{minimize}} \quad \varphi_i(x_i; x^{-i}) \\ & \text{subject to} \quad x_i \in K_i(x^{-i}). \end{array}$$

Here, for each $i \in \mathcal{N}$, the set-valued maps $K_i : \prod_{j \neq i} \mathbb{R}^{m_j} \rightarrow 2^{\mathbb{R}^{m_i}}$ and the map $K : \mathbb{R}^m \rightarrow 2^{\mathbb{R}^m}$, are defined as

$$K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N} \quad \text{and} \quad K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}) \quad \forall x \in \mathbb{R}^m. \quad (4.1)$$

\mathbb{R}_i^m , recall is the strategy space of player i and $m = \sum m_i$.

We make the following assumptions throughout this chapter.

Assumption 4.1 *For each $i \in \mathcal{N}$, the objective function $\varphi_i \in C^1$ and $\varphi_i(x_i; x^{-i})$ is convex in x_i*

for all x^{-i} . Unless otherwise mentioned, \mathbb{C} is closed and convex set.

Under this assumption a vector $x = (x_1, \dots, x_N)$ is a GNE of the above game if and only if it solves the quasi-variational inequality $\text{QVI}(K, F)$,

Find $x \in K(x)$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in K(x),$ (QVI(K, F))
--

where

$$F(x) = \begin{pmatrix} \nabla_1 \varphi_1(x) \\ \vdots \\ \nabla_N \varphi_N(x) \end{pmatrix},$$

and $\nabla = \frac{\partial}{\partial x_i}$. Critical to all our theorems is the nature of the set-valued map K . Recall that this has been investigated in depth in Section 2.2.1. We gather the relevant results here for ease of reference.

Lemma 4.1 *Let \mathbb{C} be a closed set in \mathbb{R}^m and K be as given in (4.1). Then the following hold:*

- (a) *If $\mathbb{C} = \prod_{i \in \mathcal{N}} \mathbb{C}_i$, where $\mathbb{C}_i \subseteq \mathbb{R}^{m_i}$ for every $i \in \mathcal{N}$, are nonempty, not necessarily convex sets, then $K(x) = \mathbb{C}$ for every x in \mathbb{C} and is empty otherwise.*
- (b) *For any \mathbb{C} , not necessarily convex, x is a fixed point of K if and only if $x \in \mathbb{C}$.*
- (c) *If \mathbb{C} is closed and convex, $K(x)$ is closed and convex for any $x \in \text{dom}(K)$.*
- (d) *A point x belongs to the interior of $K(x)$ if and only if x is in the interior of \mathbb{C} .*

The rest of this chapter is organised as follows. In Section 4.2, we examine Harker's conditions in depth and clarifies their shortcomings. Section 4.3 discusses the how application of Harker's result to a modified QVI avoids these shortcomings. Additionally, we show how Facchinei et al.'s result may be obtained through this application. Some concluding remarks are provided in Section 4.4.

■ 4.2 Harker's conditions

We may better understand Harker's result by seeing it within the history of results on QVIs. The QVI was first introduced by Bensoussan, Goursat and Lions [BGL73] and is a significantly

4.2 Harker's conditions

harder problem than the VI. Most of the challenges in the analysis of the QVI can be traced to the set-valued map in its definition. Therefore, many of first existence results on QVIs rested on the well-behavedness of this map, as evidenced, for e.g., by the results of Ichiishi [Ich83] and Chan and Pang [CP82] which rely on the continuity of the set-valued map and compactness of its graph. Unfortunately, continuity is too stringent a requirement that is seen to fail for many simple settings. In [Har91], Harker gave an important result that circumvents the need for continuity by constructing a VI whose solutions solve the QVI. Since numerous existence results were known for VIs, these could potentially be leveraged to claim the existence of a solution to the QVI without invoking the continuity of the set-valued map. We begin our discussion by reproducing² Theorem 3 from Harker's paper [Har91].

Theorem 4.2 (Harker [Har91]) *Let F and L be respectively point-to-point and point-to-set mappings from \mathbb{R}^m to itself. Suppose that there exists a nonempty closed, convex set A such that*

- (i) $L(x) \subseteq A$ for all $x \in A$ and
- (ii) $x \in L(x)$ for all $x \in A$.

Then any solution to the variational inequality $\text{VI}(A, F)$ is a solution of $\text{QVI}(L, F)$

It is easy to see what Harker is attempting to do: in (ii) he seeks that all points in the set A are feasible for the QVI and in (i) he requires that A be large enough to subsume the moving set $L(x)$ for each x in A . The continuity of L does not appear in the picture.

Theorem 4.2 is not limited to QVIs arising from generalized Nash games with *shared constraints*. i.e. L , in Theorem 4.2, is not required to be of the form specified in (4.1) for K . However, problems emerge when one tries to apply it with $L = K$. Consider the following: any set A satisfying (ii) must be a subset of \mathbb{C} . But combining this with (i) implies that $K(x)$ must be a subset of \mathbb{C} for all $x \in A$. In the trivial setting where \mathbb{C} is rectangular (or cartesian, cf. Lemma 4.1(a)), $K(\cdot) \equiv \mathbb{C}$ and (i) and (ii) obviously hold. But when \mathbb{C} is not rectangular, $K(x)$ appears to routinely include points outside \mathbb{C} for any x in \mathbb{C} . For e.g., in Fig. 4.1, for a given $x \in \mathbb{C}$, one of the corners of $K(x)$ lies outside \mathbb{C} . Even in the simple case where \mathbb{C} is polyhedral, K does not stay confined within \mathbb{C} .

²Theorem 4.2 is a slight modification Harker's Theorem 3 in [Har91]. Harker's version also requires that the set ' A ' in Theorem 4.2 to be compact, whereas we have required only closedness on A . Compactness is imposed presumably in keeping with the assumption prevailing in [Har91] that each player has compact strategy sets. It is trivial to check that the result is valid even under the closedness A .

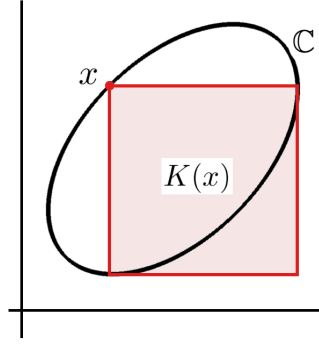


Figure 4.1: A set \mathbb{C} , x on the boundary of \mathbb{C} and $K(x)$.

This was also observed by Facchinei et al. [FFP07] through the following example.

Example 4.1. (Facchinei et al. [FFP07]) Suppose \mathbb{C} is defined as

$$\mathbb{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 \leq 2, x_2 - x_1 \geq 0, x_1, x_2 \geq 0\}.$$

Then \mathbb{C} is a triangle in \mathbb{R}^2 with vertices $(0, 0)$, $(1, 1)$ and $(0, 2)$. Let $A = \mathbb{C}$ and consider $(0, 1) \in \mathbb{C}$. Then, we have

$$K(0, 1) = [0, 1] \times [0, 2].$$

Clearly, $K(0, 1)$ is a rectangle that includes in it the triangle \mathbb{C} , implying that Theorem 4.2 (i) is not satisfied. \square

Consequently, when $L = K$, finding a set A satisfying (i) and (ii) proves extraordinarily difficult and one is led to the question of whether there exists *any* nontrivial \mathbb{C} for which Theorem 4.2 can be applied with this L .

The goal of this section is to show that if one also requires A to have a interior, the trivial setting where \mathbb{C} is rectangular is indeed the only setting where Theorem 4.2 applies with $L = K$. We show this through a combination of two results, the first of which is Theorem 4.3. Theorem 4.3 says that when $L = K$, there is at most one closed convex set A that satisfies (i), (ii) *and* has a nonempty interior and this set has to be \mathbb{C} itself. The second result is Theorem 4.4 which says that if \mathbb{C} has a nonempty interior and satisfies (i), (ii) with $L = K$, then \mathbb{C} has to be rectangular.

Theorem 4.3 *Let $\mathbb{C} \subseteq \mathbb{R}^m$ be closed and convex with nonempty interior and K be as defined in*

4.2 Harker's conditions

(4.1). Let $A \subseteq \mathbb{R}^m$ and consider the following three conditions:

- (a) A is closed, convex, and has nonempty interior
- (b) $K(x) \subseteq A$ for all $x \in A$
- (c) $x \in K(x)$ for all $x \in A$.

If A satisfies (a), (b), and (c), then $A = \mathbb{C}$. Therefore,

1. If \mathbb{C} satisfies the above three conditions, it is the only set satisfying these conditions.
2. If \mathbb{C} does not satisfy these conditions, there does not exist any set that satisfies them.

Proof : Let A satisfy conditions (a), (b), (c). If $A = \mathbb{C}$, it is easy to see that claims (1) and (2) follow logically. So we will prove $A = \mathbb{C}$.

(c) says that every point in A is a fixed point of K . So by Lemma 4.1(b) we must have $A \subseteq \mathbb{C}$. It follows that $\text{int}(A) \subseteq \text{int}(\mathbb{C})$. We will first show that $\partial A \cap \text{int}(\mathbb{C}) = \emptyset$ and use that to conclude $A = \mathbb{C}$. If $\partial A \cap \text{int}(\mathbb{C}) \neq \emptyset$, we can pick any $x \in \text{int}(\mathbb{C}) \cap \partial A$ and construct a ball B around x such that $B \subseteq \text{int}(\mathbb{C}) \cap \text{int}(K(x))$ (this is ensured by Lemma 4.1(d)). But using (b), we see that B , which is a subset of $K(x)$, has to be included in A . Since a ball around x is included in A , this means that x is in the interior of A . But this contradicts the assumption that x lies on the boundary of A . So $\partial A \cap \text{int}(\mathbb{C})$ must be empty and that ∂A must be included in $\partial \mathbb{C}$.

So we have $A \subseteq \mathbb{C}$ with $\text{int}(A) \subseteq \text{int}(\mathbb{C})$ and $\partial A \subseteq \partial \mathbb{C}$. We now show that this implies $A = \mathbb{C}$. The argument goes via contradiction. Suppose that there is a point $y \in \mathbb{C}$ which is not in A . Take some other point $z \in \text{int}(A) \subseteq \text{int}(\mathbb{C})$ and consider the segment joining y and z . It is known (see [Roc97, Theorem 6.1]) that the points $\{ty + (1-t)z, t \in [0, 1]\}$ lie in $\text{int}(\mathbb{C})$. If we can show that the segment $\{ty + (1-t)z \mid t \in (0, 1)\}$ crosses the boundary³ of A , it will contradict $\partial A \cap \text{int}(\mathbb{C}) = \emptyset$ and complete the proof. Suppose the set $\{ty + (1-t)z \mid t \in (0, 1)\}$ does not cross the boundary of A . Then the interval $(0, 1)$ is the disjoint union of the following sets

$$T_0 = \{t \in (0, 1) \mid ty + (1-t)z \in \text{int}(A)\} \quad \text{and} \quad T_1 = \{t \in (0, 1) \mid ty + (1-t)z \in A^c\}.$$

³While this may seem intuitively obvious, it is nontrivial. In the most general setting the result is called “Jordan separation theorem” [FG95]. We have given an argument based on completeness of reals, inspired by [Gli01, page 38–40].

Recall that $z \in \text{int}(A)$ and $y \in A^c$. Since A is convex, $\text{int}(A)$ is convex. So T_0 is a convex subset of $(0, 1)$. Furthermore, T_0 is open at 0 and open at the other end. i.e. $T_0 = (0, \alpha)$ for some $\alpha \in (0, 1)$. Consequently $T_1 = [\alpha, 1)$. But since T_1 is open at both ends, this is a contradiction. So the segment $\{ty + (1-t)z \mid t \in (0, 1)\}$ intersects the boundary of A . This contradicts our conclusion that $\text{int}(\mathbb{C})$ and the ∂A are disjoint. So our assumption that there exists a $y \in \mathbb{C} \cap A^c$ is incorrect, and we must indeed have $A = \mathbb{C}$. ■

Therefore, if we apply Theorem 4.2 with $L = K$ by also requiring that the set A have a nonempty interior, then there is at most one choice for A : \mathbb{C} itself. Consequently, if \mathbb{C} does not satisfy Theorem 4.2(i),(ii), it is not possible to find an A with nonempty interior satisfying Theorem 4.2(i),(ii). Next, one may ask what kind of a set may \mathbb{C} be so as to satisfy conditions (a), (b) and (c) in Theorem 4.3. The following theorem shows if \mathbb{C} satisfies these conditions then \mathbb{C} has to be rectangular. This is an important finding in that we can essentially claim that Harker's result cannot be applied to many non-trivial shared-constraint games.

Theorem 4.4 *Suppose $\mathbb{C} \subseteq \mathbb{R}^m$ and let K be as defined in (4.1). If $A = \mathbb{C}$ satisfies (a), (b), (c) from Theorem 4.3, then there exist for each $i \in \mathcal{N}$, sets $\mathbb{C}_i \subseteq \mathbb{R}^{m_i}$ such that $\mathbb{C} = \prod_{i \in \mathcal{N}} \mathbb{C}_i$.*

Proof : It suffices to show that $\text{int}(\mathbb{C})$ is rectangular. We show this through a series of steps that follow the following argument. The first step claims that the union $\bigcup_{x \in \text{int}(\mathbb{C})} \text{int}(K(x))$ is the set $\text{int}(\mathbb{C})$. Steps 2,3 together show that K is locally constant on $\text{int}(\mathbb{C})$. In Step 4, using the convexity of $\text{int}(\mathbb{C})$ we show that K is in fact constant on $\text{int}(\mathbb{C})$. Combining this with the first step gives the result.

Step 1. $\bigcup_{x \in \text{int}(\mathbb{C})} \text{int}(K(x)) = \text{int}(\mathbb{C})$.

Proof: From (b) in Theorem 4.3, $\text{int}(K(x)) \subseteq \text{int}(\mathbb{C})$ and thus $\bigcup_{x \in \text{int}(\mathbb{C})} \text{int}(K(x)) \subseteq \text{int}(\mathbb{C})$. But from Lemma 4.1(d), we get $\text{int}(\mathbb{C}) \subseteq \bigcup_{x \in \text{int}(\mathbb{C})} \text{int}(K(x))$, whereby the claim follows.

Step 2. If $x, y \in \mathbb{C}$, then $y \in K(x) \iff x \in K(y)$.

Proof: Let $y \in K(x)$. It follows that $y_i \in K_i(x^{-i})$ and $x_i \in K_i(x^{-i})$ for all i . Therefore for all i , $(x_i, y^{-i}) \in K(x)$. But Theorem 4.3 implies that if $(x_i, y^{-i}) \in K(x)$ then $(x_i, y^{-i}) \in \mathbb{C}$. Therefore by (4.1), $x \in K(y)$. Since x and y are arbitrary, the equivalence $y \in K(x) \iff x \in K(y)$ follows.

4.2 Harker's conditions

Step 3. If $x, y \in \mathbb{C}$ and $y \in K(x)$, then $K(y) = K(x)$.

Proof: As a consequence of Step 2, we have that if $y \in K(x)$, then $x \in K(y)$. We first proceed to show that $K(x) \subseteq K(y)$. Let z be a point in $K(x)$. For each i , $(z_i, y^{-i}) \in K(x)$. Arguing as in Step 2, (b) implies that $z \in K(y)$, whereby $K(x) \subseteq K(y)$. Proceeding in an identical fashion from $x \in K(y)$, we have that $K(y) \subseteq K(x)$. It follows that $K(y) = K(x)$.

Step 4. K is constant on $\text{int}(\mathbb{C})$.

Proof: We proceed to show this result by first showing that K is constant over a line segment $[x, y]$ where $x, y \in \text{int}(\mathbb{C})$. Since \mathbb{C} is convex the segment joining x and y , $[x, y]$ lies in the interior of \mathbb{C} . For any $z \in [x, y]$, let B_z be an open ball contained in $\mathbb{C} \cap K(z)$ (cf Lemma 4.1(d)). It follows that

$$\mathcal{V} := \{B_z \mid z \in [x, y]\},$$

is an open cover of $[x, y]$. Since $[x, y]$ is compact, there exists a finite subcollection, \mathcal{U} , of \mathcal{V} such that

$$[x, y] \subseteq \bigcup_{B \in \mathcal{U}} B.$$

We number balls in \mathcal{U} inductively such that at any stage of numbering, K is constant over the union of the numbered balls. Since \mathcal{U} has finitely many balls, this process will end with K being constant over \mathcal{U} . To begin, choose an arbitrary ball from \mathcal{U} and call it B_1 . By Step 3, K is constant on B_1 with value $K(z^1)$, where z^1 is such that $B_{z^1} = B_1$. Now suppose that ℓ balls have been numbered, where $1 \leq \ell \leq |\mathcal{U}|$ and K is constant over $\bigcup_{j=1}^{\ell} B_j$ and let $\bar{\mathcal{U}}_{\ell}$ be the set of balls that are yet to be numbered. The unions $\bigcup_{j=1}^{\ell} B_j$ and $\bigcup_{B \in \bar{\mathcal{U}}_{\ell}} B$ are open sets. If these unions are disjoint, we would get that $[x, y]$ is disconnected, which is absurd, since $[x, y]$ is convex. Therefore there exists a ball $B_{\ell+1}$ where in $\bar{\mathcal{U}}_{\ell}$ such that $B_{\ell+1} \cap \bigcup_{j=1}^{\ell} B_j \neq \emptyset$. It follows that K is constant over $B_1, \dots, B_{\ell+1}$. Continuing in this manner we get K is constant over $\bigcup_{B \in \mathcal{U}} B$, and in particular, over $[x, y]$. Since x and y were arbitrary points in $\text{int}(\mathbb{C})$, we get K is constant over $\text{int}(\mathbb{C})$.

Step 5. $\text{int}(\mathbb{C})$ is rectangular.

Proof: Since K is constant over $\text{int}(\mathbb{C})$, the conclusion of Step 1 degenerates to $\text{int}(\mathbb{C}) = \text{int}(K(x))$ for each $x \in \text{int}(\mathbb{C})$. But since K is rectangular, $\text{int}(\mathbb{C})$ is rectangular. ■

We may summarize the conclusions of Theorems 4.3 and 4.4 as follows. When $L = K$, Theorem 4.3 limits our choice for sets A with nonempty interior that satisfy Harker's conditions to only the set \mathbb{C} . But Theorem 4.4 goes to show that \mathbb{C} cannot be a choice for A for verifying Harker's conditions with $L = K$ unless \mathbb{C} is rectangular. But in this case, \mathbb{C} isn't really a *shared* constraint at all and the $\text{QVI}(K, F)$ is the same as $\text{VI}(\mathbb{C}, F)$. It is therefore safe to conclude that taking $L = K$ is not the best way to apply Theorem 4.2 to shared-constraint games. Section 4.3 presents an alternative way of applying Theorem 4.2.

Some technical remarks are worth mentioning at this juncture regarding the above proof. First, it is well known (see, e.g., p. 114, [Kum05]) that a locally constant function on a connected space is constant. Presumably, such a result also holds for set-valued maps and Step 4 could potentially be claimed through this result. Unfortunately, we are not aware of any such result.

Secondly, we note some caveats about Theorem 4.3 and Theorem 4.4. That the set A sought by Theorem 4.3 must have an interior is an important ingredient for the validity of Theorem 4.3. The argument, that any line joining a point inside A to a point outside A must cross the boundary, used in proving Theorem 4.3 holds only when A has an interior. In fact we have an example in [KS10b] of an A without an interior that satisfies Harker's conditions for a specific \mathbb{C} . The set \mathbb{C} is such a point $x^* \in \partial\mathbb{C}$ exists, for which $K(x^*)$ is the singleton $\{x^*\}$. Clearly one may take $A = \{x^*\}$ to verify Harker's conditions. We have reproduced the relevant portions of the example here.

Example 4.2. Fig. 4.2 shows $\mathbb{C} \subseteq \mathbb{R}^2$ and a point $x^* \in \partial\mathbb{C}$ with the property that the image of x^* under K is a singleton, namely x^* itself. In Fig. 4.2, dotted lines depict axes with their origin shifted to x^* . If $y \in K(x^*)$, the points (y_1, x_2^*) and (x_1^*, y_2) lie in on these 'axes'. Notice that since these 'axes' intersect \mathbb{C} at only one point, x^* , $K(x^*) = \{x^*\}$. □

The nonemptiness of $\text{int}(\mathbb{C})$ is also essential to Theorem 4.4, as seen in the following example.

Example 4.3. Let $\mathbb{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1 = x_2)\}$. Suppose \mathbb{C} is a line in \mathbb{R}^2 and has no interior. For any $x \in \mathbb{C}$, we have $K(x) = \{x\}$. Clearly $A = \mathbb{C}$ satisfies (b) and (c) from Theorem 4.3, but it does not satisfy (a). □

4.3 Applying Harker's conditions to the modified QVI

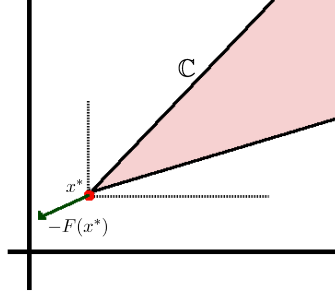


Figure 4.2: An example where $K(x^*) = \{x^*\}$.

■ 4.3 Applying Harker's conditions to the modified QVI

Facchinei et al. have rightly observed in [FFP07, section 4] that Harker's result is hard to use if $L = K$. In [FFP07, p. 162], the authors state “*The problem with this (Harker's) result is, we believe, that it is not simple to give classes of problems for which (i) and (ii) are satisfied for some easily calculated A . Rosen's setting (i.e. generalized Nash games with shared constraints) analyzed in this paper is certainly not covered (when $A = \mathbb{C}$, as suggested in Harker's paper for polyhedral \mathbb{C}). In fact, in Rosen's setting, condition (ii) is obviously always satisfied, but condition (i) need not, except when \mathbb{C} is a rectangle.*” and provide Example 4.1 to illustrate this difficulty.

As an alternative to Harker's result, Facchinei et al. gave a result that prescribed a VI associated with a shared-constraint game that provided a solution to the original QVI ([FFP07, Theorem 2.1]). We present this result next.

Theorem 4.5 (Facchinei et al. [FFP07]) *Let F be continuous and K be as defined by (4.1). Then every solution of $\text{VI}(\mathbb{C}, F)$ is a solution of $\text{QVI}(K, F)$.*

Unlike Harker's result, this result is more direct and clearly applies to all shared-constraint games.

It is evident from [FFP07] and Example 4.1 that the observations made by Facchinei et al. were based on taking L to be the map K defined in (4.1). Indeed, our results from Section 4.2 formalize and substantiate their observations. But despite the results from Section 4.2, Harker's result is still of relevance to shared-constraint games. Our contention is that the difficulty of applying Harker's result arises from taking $L = K$ and in assuming that this is the *only* way to apply Harker's result. Harker's result is better applied to a modified $\text{QVI}(K \cap \mathbb{C}, F)$ as opposed to a direct application to

$\text{QVI}(K, F)$. When applied in this manner, Harker's result *implies* the Theorem 4.5 of Facchinei et al. We elaborate on this in this section. Consider $\text{QVI}(K \cap \mathbb{C}, F)$ defined as

Find $x \in K(x) \cap \mathbb{C}$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in K(x) \cap \mathbb{C}.$	$\text{QVI}(K \cap \mathbb{C}, F)$
--	------------------------------------

This seemingly weaker QVI is in fact equivalent to $\text{QVI}(K, F)$.

Proposition 4.6 *Let F be continuous and K be as defined by (4.1). Then, we have*

$$\text{SOL}(\text{QVI}(K, F)) = \text{SOL}(\text{QVI}(K \cap \mathbb{C}, F)).$$

Proof : “ \subseteq ” If x solves $\text{QVI}(K, F)$, we have $x \in K(x)$ and $F(x)^T(y - x) \geq 0 \quad \forall y \in K(x)$. It follows that $x \in \mathbb{C}$ and hence $x \in K(x) \cap \mathbb{C}$. It is easy to see that $F(x)^T(y - x) \geq 0$ for every $y \in K(x) \cap \mathbb{C}$. Thus x solves $\text{QVI}(K \cap \mathbb{C}, F)$.

“ \supseteq ” Let $x \in K(x) \cap \mathbb{C}$ solve $\text{QVI}(K \cap \mathbb{C}, F)$ and let $y \in K(x)$ be arbitrary. By definition in (4.1), for every $i \in \mathcal{N}$, the point $(y_i, x^{-i}) \in \mathbb{C}$. Furthermore, since $x \in K(x)$, the point (y_i, x^{-i}) also belongs to $K(x)$ for each $i \in \mathcal{N}$. Thus for each $i \in \mathcal{N}$, $(y_i, x^{-i}) \in K(x) \cap \mathbb{C}$. Since x solves $\text{QVI}(K \cap \mathbb{C}, F)$, we have

$$F(x)^T((y_i, x^{-i}) - x) \geq 0, \quad \text{i.e.,} \quad F_i(x)^T(y_i - x_i) \geq 0, \quad (4.2)$$

for each $i \in \mathcal{N}$. Summing (4.2) over all $i \in \mathcal{N}$ we get $F(x)^T(y - x) \geq 0$. Since y was an arbitrary point in $K(x)$, x solves $\text{QVI}(K, F)$. ■

It is easy to see the issue arising in Example 4.1 does not arise when $L(\cdot) \equiv K(\cdot) \cap \mathbb{C}$. Indeed $L(\cdot) = K(\cdot) \cap \mathbb{C}$ satisfies Harker's conditions, i.e. (i), (ii) from Theorem 4.2, with $A = \mathbb{C}$. Thus for shared-constraint games Theorem 4.2 *always* applies to $\text{QVI}(K \cap \mathbb{C}, F)$. And when Theorem 4.2 is applied with $L(\cdot) = K(\cdot) \cap \mathbb{C}$, one obtains immediately Theorem 4.5 of Facchinei et al..

Theorem 4.7 *Let F be continuous and K be as defined by (4.1). Then every solution of $\text{VI}(\mathbb{C}, F)$ is a solution of $\text{QVI}(K, F)$.*

Proof : In Theorem 4.2, take $L(\cdot) := K(\cdot) \cap \mathbb{C}$. It is not hard to see that $A = \mathbb{C}$ satisfies conditions

4.4 Conclusions

Theorem 4.2(i),(ii) with this L :

- (i) $K(x) \cap \mathbb{C} \subseteq \mathbb{C}$ for all $x \in \mathbb{C}$
- (ii) $x \in K(x) \cap \mathbb{C}$ for all $x \in \mathbb{C}$ (cf. Lemma 4.1(b))

Now applying Theorem 4.2 gives

$$\text{SOL}(\text{VI}(\mathbb{C}, F)) = \text{SOL}(\text{VI}(A, F)) \subseteq \text{SOL}(\text{QVI}(L, F)) = \text{SOL}(\text{QVI}(K \cap \mathbb{C}, F)).$$

But by Proposition 4.6, $\text{SOL}(\text{QVI}(K \cap \mathbb{C}, F)) = \text{SOL}(\text{QVI}(K, F))$. Consequently

$$\text{SOL}(\text{VI}(\mathbb{C}, F)) \subseteq \text{SOL}(\text{QVI}(K, F)),$$

which is the result of Theorem 4.5. ■

In other words Harker's approach, with an application to $\text{QVI}(K \cap \mathbb{C}, F)$, in fact subsumes the approach of Facchinei et al.. The interested reader may also check that Theorem 4.3 applies even with ' $K(x)$ ' replaced by ' $K(x) \cap \mathbb{C}$ ' in Theorem 4.3(b),(c). (The proof requires an exact repetition of the steps used to prove Theorem 4.3.) Thus for the $\text{QVI}(K \cap \mathbb{C}, F)$ too, the only set with a nonempty interior satisfying Harker's conditions is \mathbb{C} itself.

■ 4.4 Conclusions

Equilibria of generalized Nash games are wholly captured by quasi-variational inequalities. However, such objects are analytically less tractable. Accordingly, Harker [Har91] presented a result which showed that a suitable variational inequality gives a solution to the original QVI. Unfortunately, the direct application of this result to the class of shared-constraint Nash games proves challenging for its hypothesis are hard to satisfy. We examine this issue more carefully and show that under mild assumptions, it is, in fact, impossible to satisfy the hypothesis of Harker's result, when it is applied in a direct manner. In fact, we provide a formal result to support observations made by Facchinei et al. [FFP07]; in particular, they suggested that Harker's result was only applicable to shared-constraint Nash games when \mathbb{C} was rectangular.

We proceed to show that Harker's result can indeed be applied to this class of games, albeit in an indirect manner, by an application to a modified QVI for which the hypothesis are always satisfied. The modified QVI has the same solution set as the original QVI and allows us to independently derive an analogous result of Facchinei et al. [FFP07].

Chapter 5

The Efficiency of Generalized Nash Equilibria

■ 5.1 Introduction

Generalized Nash games with shared constraints are noncooperative N -player games in which the strategies available to each player are constrained by the requirement that the tuple of player-strategies belong to a common set. This work considers shared-constraint games which arise from the competition for a finite resource, such as bandwidth or energy. The shared constraint is the common requirement that the portions of the resource allocated to all players must total to not exceed the available quantity of the resource.

Our game may be thought of as a *resource allocation game*, though this term has acquired a somewhat more specific meaning that does not commonly allude to our setting [JT04]. The traditional resource allocation game has players that compete in a noncooperative fashion to access the resource, but their competition is induced by a *mechanism*. A mechanism accepts bids from players and allocates the resource to the players in exchange for a payment, thereby also maintaining feasibility of the allocation. Since players may be price anticipating, the choice of a player's bid is strategically coupled to the bids of other players, leading to a game in which the strategy of each player is its bid, its constraint is the space of its bids and its objective is to maximize its utility less the payment charged by the mechanism. In comparison, our game is played in the space of resource portions, the players are cognizant of the finiteness of the resource and their objective is to maximize their utility.

The goals of resource allocation are two-fold, one from the perspective of the players demanding the resources and the other from the perspective of the system or society as a whole. From the player-perspective, each player desires to maximize its utility. A common way of satisfying this goal is by introducing competition amongst the players, such as through an auction or a mechanism as above, and thereby creating a resource allocation *game*. From the system-perspective, the goal is to allocate resources in a manner that a prescribed societal objective, characterized by the optimization of a *social choice function*, is met. The optimizing allocation is said to *implement* the social choice function. A natural and desirable societal objective is the optimization of *aggregate* utility or sum of utilities of all players, i.e. the maximization of social welfare, though other objectives could be chosen based on the setting. When utilities are measured in monetary terms and arbitrary money transfers are permitted across agents, aggregate utility maximization also equivalent to the classical concept of *Pareto efficiency*¹ [Joh04] or *economic efficiency*. We say that an allocation is *efficient* if it implements the aggregate utility and we call the *efficiency* of an allocation, the ratio of the aggregate utility for this allocation to that of an efficient allocation.

The allocation corresponding to the Nash equilibrium of the game is, in general, only feasible for the social planner's problem of maximizing aggregate utility. Thus allocations that result from a competition for the resource are, in general, not efficient. The question of resource allocation is therefore one with dual, and possibly conflicting objectives of efficiency and competition. The usual approach to this dilemma gives primacy to competition while attempting to control the loss in efficiency. Specifically, when allocating resources through a mechanism, the effort is to design a mechanism that provides a guarantee of low efficiency loss at its Nash equilibrium. This is the central pursuit of the field of *mechanism design* [NGNP09], though the societal objective to be met differs according to situation.

Our game, though dealing with allocation of resources, represents a departure from this approach. Though we also have the dual objectives of competition and efficiency, our game is relevant if one makes the assumption that the option of a mechanism does not exist. In our setting, players move simultaneously, in a noncooperative manner and compete for portions of the resource. A shared-constraint game naturally fits such an interaction. The goal behind our work is not to suggest

¹A Pareto efficient allocation is an allocation, by deviations from which the value derived by any player cannot be strictly increased without simultaneously resulting in strictly lower value for another player.

5.1 Introduction

this game as an alternative to mechanism driven approaches to resource allocation, but instead to analyze the specific setting where it is relevant and to study the efficiency of equilibria that result from it. The subtlety though is that shared-constraint games admit two kinds of solution concepts, generalized Nash equilibrium (GNE) and the variational equilibrium (VE), with two different economic interpretations. In the light of this, our goal is also to do a comparative study of these equilibria with respect to the metric of efficiency. Though our work is not intended as an alternative to other mechanism-based approaches, it contributes to identifying settings where the VEs of this shared-constraint game have higher efficiency than the equilibria induced by mechanisms.

The distinction between the GNE and the VE and was discussed in detail in Chapter 2. We discuss the efficiency of these equilibria separately. In particular, we are concerned with the best case efficiency and the worst case efficiency of the GNE and the VE. Ours is, to the best of our knowledge, the only work on the efficiency of equilibria in general shared-constraint games, though the social welfare of the equilibria of these games has been considered in the setting of congestion control [AB02]. A detailed comparison with other is encompassed in Section 5.3. We consider two kinds of utility functions. In the first kind, the utility derived by any player is a function only of the allocation it receives. In the second, and more general setting, utilities are dependent on allocations received by other players. The first setting has an interesting interpretation in terms of the *competitive* equilibrium. We also see that the VE is efficient in this setting, whereas the GNE can be arbitrarily inefficient.

In the more general setting we are concerned with the best case efficiency and worst case efficiency over a class of utility functions. We characterize settings under which full efficiency is obtained and identify that the solution concept of the VE achieves this unit efficiency. We also show that a departure from this setting can lead to arbitrarily low efficiency. Specifically, if one considers the GNE as a solution concept, for settings where the VE is efficient, one can get arbitrarily low efficiency. And a departure from the “efficient” setting of the VE can lead to arbitrarily low efficiency for the VE.

We then suggest some ways in which this arbitrarily low efficiency may be remedied. One of the causes of low efficiency is the possibility of wide variation in the marginal utilities of players. A more restricted class of utility functions in which the gradient map of every member function is bounded away from zero and from above uniformly over the domain, gives a more favorable worst

case efficiency. The choice of the social choice function is not universal and notions of efficiency with respect to other system problems may be more relevant in other settings. We consider one such setting, inspired by [AB02]. This setting is a game where players incur costs that, from the system point of view are not additive. Thus the system problem is not merely the sum of the objectives of all players. We characterize utility functions for which the VE is efficient under this notion of efficiency. Finally we consider the imposition of a reserve price on players. The reserve price has the effect of eliminating players with low interest in the resource. The GNE of the resulting game is more indicative of the system optimal solution. We find that under certain conditions, efficiency as high as unity is obtainable by the imposition of an appropriate reserve price.

In Section 5.2 we introduce the model and our assumptions. In Section 5.3 we clarify the relationship of our approach with the competitive equilibrium and other approaches that use mechanisms. Following that we give a general bound on the efficiency by showing that In Section 5.5 we characterize the class of games where the VE is efficient and show that the GNE can be arbitrarily inefficient in this class. Section 5.6 some ways to lower bound the efficiency loss and some other models are considered. We conclude with some final considerations in Section 5.7.

■ 5.2 Preliminaries

In this section we describe the setting for our resource allocation game and our notions of efficiency. We mention some mathematical characterizations that will be used in later sections.

We first describe the game abstractly and then explain the terms specific to resource allocation. Let $\mathcal{N} = \{1, 2, \dots, N\}$ be a set of players. For each $i \in \mathcal{N}$, let $x_i \in \mathbb{R}$ be player i 's strategy and $\varphi_i : \mathbb{R}^N \rightarrow \mathbb{R}$ be his utility function. As before, by x we denote the tuple (x_1, x_2, \dots, x_N) , x^{-i} denotes the tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ and (y_i, x^{-i}) the tuple $(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_N)$. The *shared constraint* in our game is the requirement that the tuple x be constrained to lie in a set $\mathbb{C} \subseteq \mathbb{R}^N$,

$$\mathbb{C} = \left\{ x \geq 0 \mid \sum_{j \in \mathcal{N}} x_j \leq C \right\},$$

where C is a positive real number

A tuple $x \in \mathbb{C}$ is termed an *allocation* and the number C represents the capacity of the resource.

5.2 Preliminaries

We consider an model with a single resource, whereby x_i is the portion of the resource demanded by player i . In the generalized Nash game with shared constraint \mathbb{C} , player i is assumed to solve the parameterized optimization problem,

$$\begin{array}{ll} A_i(x^{-i}) & \text{maximize}_{x_i} \quad \varphi_i(x_i; x^{-i}) \\ & \text{subject to} \quad \sum_{j \in \mathcal{N}} x_j \leq C, \\ & \quad \quad \quad x_i \geq 0. \end{array}$$

In other words, each player decides a portion of the resource so as to maximize his utility, but he does so while being cognizant of the fact that portion he has access to is that which when added to the portions of other players, does not exceed C . Throughout this chapter we abbreviate the aggregate utility function by

$$\theta \triangleq \sum_{j \in \mathcal{N}} \varphi_j, \tag{5.1}$$

and make the following assumption.

Assumption 5.1 *We assume that for each $i \in \mathcal{N}$, the utility function $\varphi_i(x)$ is a concave, continuously differentiable and strictly increasing function in x_i . Furthermore, the utility obtained from the allocation $x = 0$, i.e. $\varphi(0)$, is nonnegative. Finally, the aggregate utility θ is a concave function such that for each allocation x , $\nabla \theta(x)$ is nonnegative.*

Following are the motivations behind these assumptions. It is common to assume that utility is an increasing function of the portion, and that *marginal utility* is nonincreasing. This leads one to an assumption that of each φ_i is increasing and strictly concave in x_i . The other assumptions, in particular the concavity of θ and nonnegativity of $\nabla \theta$ may not hold in all cases. The nonnegativity of $\nabla \theta$ says that utility functions of players are such that for the social planner, withholding portions of the resource so as to not exhaust the capacity C , is not optimal. The concavity of θ is a technical assumption that helps simplify our analysis. We note that these assumptions are compatible with the usual case where for all i , $\varphi_i(x) = U_i(x_i)$ for all x and that they are met by quasi-linear utility functions: $\varphi_i(x) = U_i(x_i) - \sum_{j \neq i} d_j^i x_j$ [MWG95].

Let Φ denote the tuple $(\varphi_1, \dots, \varphi_N)$. Recall the solution concepts of the generalized Nash equilibrium (GNE) and variational equilibrium (VE) from Section 1.1.1. For this game, these concepts

reduce to the following.

Definition 5.1 (Generalized Nash equilibrium (GNE)) *A tuple x is a generalized Nash equilibrium if there exists a tuple $\Lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ such that x and Λ satisfy the KKT conditions of the optimization problems $(A_1), \dots, (A_N)$:*

$$\begin{aligned} 0 &\leq x_i \perp -\nabla_i \varphi_i(x) + \lambda_i \geq 0 \\ 0 &\leq \lambda_i \perp C - \mathbf{1}^T x \geq 0, \quad \forall i \in \mathcal{N}. \end{aligned}$$

The set of all GNEs of this game with objective functions Φ is denoted by $\text{GNE}(\Phi)$.

Here $\mathbf{1}$ denotes a vector in \mathbb{R}^N with each coordinate 1. The GNE that corresponds to equal Lagrange multipliers, i.e. $\Lambda = \lambda \mathbf{1}$ for some λ is the VE.

Definition 5.2 (Variational equilibrium (VE)) *A tuple x is a variational equilibrium if there exists $\lambda \in \mathbb{R}$ such that x satisfies the KKT conditions of all the optimization problems $(A_1), \dots, (A_N)$ with λ as the Lagrange multiplier. i.e.*

$$\begin{aligned} 0 &\leq x_i \perp -\nabla_i \varphi_i(x) + \lambda \geq 0 \\ 0 &\leq \lambda \perp C - \mathbf{1}^T x \geq 0, \quad \forall i \in \mathcal{N}. \end{aligned}$$

The set of all VEs of this game with objective functions Φ is denoted as $\text{VE}(\Phi)$.

As a consequence of the compactness of \mathbb{C} , any game with utility functions satisfying Assumption 5.1, the set of VEs (and hence the set of GNEs) is nonempty. The proof of this fact can be found, e.g., in Rosen [Ros65].

The goal of this work is to study the efficiency of these solution concepts. As mentioned in the introduction, our notion of efficiency is the maximization of aggregate utility.

Definition 5.3 *A point x is said to be efficient if it solves the following optimization problem*

SYS	$\begin{aligned} &\underset{x}{\text{maximize}} && \theta(x) \\ &\text{subject to} && \sum_{j \in \mathcal{N}} x_j \leq C, \\ &&& x_j \geq 0, \quad \forall j \in \mathcal{N}. \end{aligned}$
-----	---

5.2 Preliminaries

The efficiency of a point x is defined as the ratio $\frac{\theta(x)}{\theta(x^*)}$, where x^* is a solution of (SYS).

By Assumption 5.1, (SYS) is a convex optimization problem. It follows that x is efficient if there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} 0 &\leq x \perp -\nabla\theta(x) + \lambda \mathbf{1} \geq 0 \\ 0 &\leq \lambda \perp C - \sum_{j \in \mathcal{N}} x_j \geq 0 \end{aligned}$$

An alternative characterization of the GNEs and VEs is through the use of quasi-variational inequalities and variational inequalities respectively (cf. Section A.3). To define these objects, let $F : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be the following function

$$F(x) = - \begin{pmatrix} \nabla_1 \varphi_1(x) \\ \vdots \\ \nabla_N \varphi_N(x) \end{pmatrix}, \quad \forall x \in \mathbb{R}^N,$$

and K be the set-valued map

$$K(x) := \prod_{i \in \mathcal{N}} K_i(x^{-i}) \quad \text{where} \quad K_i(x^{-i}) := \{y_i \in \mathbb{R}^{m_i} \mid (y_i, x^{-i}) \in \mathbb{C}\}, \quad \forall i \in \mathcal{N}, \forall x \in \mathbb{R}^N. \quad (5.2)$$

A allocation x is a GNE if and only if x solves the quasi-variational inequality QVI(K, F) below.

Find $x \in K(x)$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in K(x).$ (QVI(K, F))
--

Likewise, x is a VE if and only if it solves the variational inequality VI(\mathbb{C}, F).

Find $x \in \mathbb{C}$ such that $F(x)^T(y - x) \geq 0 \quad \forall y \in \mathbb{C}.$ (VI(\mathbb{C}, F))
--

Since under Assumption 5.1 (SYS) is a convex optimization problem, efficient allocations are characterized by the solutions of the VI($\mathbb{C}, -\nabla\theta$),

Find $x \in \mathbb{C}$ such that $-\nabla\theta(x)^T(y - x) \geq 0 \quad \forall y \in \mathbb{C}.$ (VI($\mathbb{C}, -\nabla\theta$))
--

Chapter 5. The Efficiency of Generalized Nash Equilibria

The efficiency of an allocation depends changes with utility functions considered. In order to provide guarantees of efficiency of an allocation one needs to consider classes of utility functions and examine the worst (or best) case of efficiency over all of them. Therefore to provide efficiency guarantees for a *solution concept*, one needs to examine the worst (or best) case of efficiency over allocations generated by the said solution concept for each utility functions in the class. For this purpose denote by \mathcal{F} the class of objective functions that satisfy Assumption 5.1:

$$\mathcal{F} = \{\Phi \mid \varphi_1, \dots, \varphi_N \text{ satisfy Assumption 5.1}\}.$$

Let \mathcal{L} be the subclass of \mathcal{F} comprising of *linear* objective functions:

$$\mathcal{L} = \{\Phi \mid \varphi_1, \dots, \varphi_N \text{ are linear and satisfy Assumption 5.1}\}.$$

Following are our notions of best case and worst case efficiency for the GNE and VE respectively.

Definition 5.4 *The best case efficiency of the GNE and the VE are defined as*

$$\begin{aligned} \bar{\rho} &= \sup_{x \in \text{GNE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)}, \\ \text{and } \bar{\vartheta} &= \sup_{x \in \text{VE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)}, \end{aligned}$$

respectively.

Definition 5.5 *The worst case efficiency of the GNE and the VE are defined as*

$$\begin{aligned} \underline{\rho} &= \inf_{x \in \text{GNE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)}, \\ \text{and } \underline{\vartheta} &= \inf_{x \in \text{VE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)}, \end{aligned}$$

respectively.

By Assumption 5.1, $\theta(0) \geq 0$ and $\nabla \theta \geq 0$. This ensures that θ is nonnegative on \mathbb{C} and thereby the solution of (SYS) is positive and finite, whereby the efficiencies defined above are all finite and nonnegative. In other words, $\bar{\rho}, \underline{\rho}, \bar{\vartheta}, \underline{\vartheta} \in [0, 1]$, and that $\bar{\rho} \geq \underline{\rho}$ and $\bar{\vartheta} \geq \underline{\vartheta}$. Furthermore, since every

5.3 Relation to past work

VE is a GNE, we readily get

$$\bar{\rho} \geq \bar{\vartheta}, \quad \text{and} \quad \underline{\rho} \leq \underline{\vartheta}.$$

■ 5.3 Relation to past work

The appropriate allocation of resources is perhaps the fundamental concern that led to the inquiries that we today recognize as being part of the field of economics. Adam Smith's classic, *The Wealth of Nations* [Smi08], first published back in 1776, contains perhaps the first scholarly attempt at understanding and explaining how societies allocate resources. Smith's observations, which say that competition and efficiency go hand in hand, are the foundations of *general equilibrium theory* and the *welfare theorems* [MWG95, Wal52]: every competitive equilibrium maximizes social welfare and under certain conditions, every social welfare maximizing allocation is achievable as a competitive equilibrium.

These theorems rely on an assumption of what is known as *perfect competition*. In perfect competition, players do not strategically anticipate the impact of their actions on the actions of other players. In particular, players take prices as fixed and make choices as though prices were given exogenously [Mas80]. Things changed in the mid-1940's with the invention of game theory [vM44]. Game theory provided a formal paradigm for studying the more general and, in some cases, somewhat more realistic setting where players were allowed to be strategic and *price anticipating*. In game-theoretic parlance, social welfare maximization was seen as the outcome of a *cooperative* game. By considering the Nash equilibrium as the outcome of the noncooperative counterpart, it was easily seen that the welfare theorems need not hold, i.e. cooperative and noncooperative games yield patently different equilibria. The classic game of *Prisoner's Dilemma* [MWG95] is a telling example of this fact. The noncooperative interaction yields both prisoners significantly more years in prison than a decision they could have achieved had they cooperated. In fact a theorem of Dubey [Dub86] rigorously establishes that Nash equilibria are generically (for an open dense subset of the space of utility functions) inefficient in the Pareto sense².

Efficiency has also been a question in mechanism design. Though Nash equilibria can be ineffi-

²See also Maskin [Mas99] which gives sufficient conditions for a social choice rule to implementable by a game and the Pigou example [NRTV07]

cient, in the context of mechanism design, these Nash equilibria correspond to a somewhat different game with different objectives for players. The efficiency, though, is measured vis-a-vis the original utility functions. When viewed in this sense, the possibility of efficiency remains open and more encouraging results are seen. The Vickrey auction [Vic61], is a success story of this approach, for it is known to be efficient³ and have other attractive properties. The last decade in particular has seen a dramatic surge in the interest in such questions, especially in the computer science and electrical engineering community. Most of this interest find its origins in the seminal work of Kelly et al. [KMT98] in which congestion or rate control in a communication network, such as the internet, was modelled as a resource allocation problem. Kelly et al. introduced a model in which each player submitted a “bid” and received an allocation proportional to its bid and showed that the competitive equilibrium of this mechanism was efficient. Since then Johari [Joh04], Roughgarden [Rou02, RT04], Papadimitriou [Pap01] and many others have worked on lower bounding the efficiency loss in resource allocation or related games. Indeed, Papadimitriou has called the worst case efficiency as the *price of anarchy*. The volume of work in this area is too large to be cited here to any degree of completeness. We instead refer the reader to the above references and to the book by Nisan et al. [NRTV07] for more.

Our work is distinct from all the other work on resource allocation. But thanks to this long and rich history of resource allocation, it does bear similarities to some of the past approaches. We highlight the distinctions here and the relation other approaches is explained in the following sections. The first distinction between our approach and any of the past approaches is the use of a shared-constraint game. Though shared-constraint game formulation appeals naturally to resource allocation, to the best of our knowledge there is no work directly shared-constraint games as a vehicle for the abstract question of resource allocation. There is however work in the specific context of congestion in networks [AB02]. Secondly, our work does not attempt to provide an alternative to mechanisms, but instead considers the setting where the option of a mechanism is not available. In our setting, players wish to split a pie N -ways without communication with each and without the intervention of an administrator while simultaneously moving to get pieces of it. This is perhaps an example of a somewhat extreme *anarchy* in resource allocation. Finally, the shared-constraint

³The Vickrey auction pertains to the auctioning of a *single* indivisible good. Efficiency in this context means that the good is allocated to the bidder who values it most.

5.3 Relation to past work

game admits to kinds of equilibria (GNE and VE) and with different consequences on efficiency. We want to know how they perform in the context of efficiency. This question has not been studied before.

The varied nature of the equilibria of the shared-constraint game has the potential to yield extremely high efficiency as well as extremely low efficiency and therein lies the possibility for similarities with other approaches. With utility functions of the “perfectly competitive” kind, i.e. the utility obtained by a player is a function only of the allocation received by it, the VE is seen to be identical to the competitive equilibrium. We elaborate on this comparison in Section 5.3.1. In Section 5.3.2, we compare our approach for this same model with a typical mechanism-based approach.

■ 5.3.1 Relation to the competitive equilibrium

Consider a setting with one *seller* and N buyers and a single resource with capacity C . Buyers are similar to what we have called *players* in this work; they are characterized by utility functions $\varphi_i(x) = U_i(x_i)$ for all x and $i \in \mathcal{N}$, which we assume satisfy Assumption 5.1. Imagine that each buyer sees a unit price p for the resource. Taking this price for granted, the buyer decides a quantity to buy so as to maximize his net payoff which is his utility less the payment he makes. Thus buyer i is faced with the following optimization problem.

B_i	$\underset{x_i}{\text{maximize}} \quad U_i(x_i) - px_i$
	$\text{subject to} \quad x_i \geq 0.$

The seller, on the other hand, seeks to maximize revenue. The seller's decision variables are the quantity sold to each buyer and he is constrained by the capacity of the resource. The seller also takes this price for granted in making his decision and is therefore faced with the following

Chapter 5. The Efficiency of Generalized Nash Equilibria

optimization problem.

S	$\begin{aligned} & \underset{x}{\text{maximize}} && p(\mathbf{1}^T x) \\ & \text{subject to} && \mathbf{1}^T x \leq C, \\ & && x \geq 0. \end{aligned}$
---	---

Now imagine a societal objective of maximizing the aggregate utility of all buyers⁴, which corresponds to the following problem.

SOC	$\begin{aligned} & \underset{x}{\text{maximize}} && \sum_{i \in \mathcal{N}} U_i(x_i) \\ & \text{subject to} && \mathbf{1}^T x \leq C, \\ & && x \geq 0. \end{aligned}$
-----	---

The setting described above is called the perfectly competitive setting. Notice that buyers and sellers *simultaneously* the quantities x_1, \dots, x_N . The “price” is assumed to be exogenously given⁵ and in addition there is a societal objective maximizing social welfare with the same decision variables. The first welfare theorem states that there exists a price p such that all the above optimization problems are consistently solved. i.e. there exists a price p and an allocation x so that $x_i \in \text{SOL}(B_i)$, for all $i \in \mathcal{N}$, $x \in \text{SOL}(S)$ and $x \in \text{SOL}(\text{SOC})$. The evidence for this lies in the fact that the KKT conditions of the problem (SOC) which say that x solves (SOC) if there exists a $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} 0 &\leq x_i \perp -\nabla_i U_i(x_i) + \lambda \geq 0, \\ 0 &\leq \lambda \perp C - \mathbf{1}^T x \geq 0, \quad \forall i \in \mathcal{N}, \end{aligned}$$

are identical to the KKT conditions of $(B_1), \dots, (B_N)$ and (S) taken together with $p = \lambda$. This x is called the *competitive equilibrium*⁶, and the first welfare theorem states that the competitive equilibrium is efficient.

⁴This is also the *aggregate surplus* of buyers and sellers. See [Joh04] for more on this.

⁵Akin to Adam Smith’s *Invisible Hand*

⁶Oftentimes the tuple x, p is called the competitive equilibrium

5.3 Relation to past work

Now consider a shared constraint game with these buyers as players. Player i solves the problem

$$\begin{array}{ll} A_i(x^{-i}) & \text{maximize } U_i(x_i) \\ & \text{subject to } \mathbf{1}^T x \leq C, \\ & x_i \geq 0. \end{array}$$

Consider the solution concept of the VE. The allocation x is a VE for this game if and only if there exists λ such that

$$\begin{aligned} 0 \leq x_i \perp -\nabla U_i(x_i) + \lambda &\geq 0, \\ 0 \leq \lambda \perp C - \mathbf{1}^T x &\geq 0, \quad \forall i \in \mathcal{N}. \end{aligned}$$

Herein lies the connection between the competitive equilibrium and our model. For utility functions of the kind considered in perfectly competitive settings, the solution concept of the VE provides an allocation identical to that of a competitive equilibrium and is thereby efficient. Thus one may say that at a VE, players demand resources *as if* they were sold by a seller.

■ 5.3.2 Relation to mechanism design

Resource allocation through a shared-constraint game is different from that through the use of mechanisms. To clarify this difference we compare our approach with the approach of Johari [Joh04, JT04], which may be taken to be a canonical mechanism-based approach to resource allocation.

Example 5.1. Johari considers the utility functions from the perfectly competitive setting $\varphi_i(x) = U_i(x_i)$ for all x . Every player submits a *bid*, or willingness to pay to a system administrator. Let the bids of the players be w_1, \dots, w_N where each $w_i \in [0, \infty)$. The system administrator aggregates these bids and allocates a portion of the resource according to an allocation rule. For e.g., Johari uses the *proportional* allocation rule wherein player i gets a portion x_i

$$x_i = \frac{w_i}{\sum_{j \in \mathcal{N}} w_j} C.$$

He is charged a payment w_i . The price for a unit of the resource is $\sum_{j \in \mathcal{N}} w_j / C$. When players

Chapter 5. The Efficiency of Generalized Nash Equilibria

are price taking they choose their bids taking $p = \sum_{j \in \mathcal{N}} w_j / C$ as constant and receive a quantity $x_i = w_i / p$. It is easy to show that this is allocation of the competitive equilibrium [Kel97].

When players are price-anticipating, i.e. they strategically anticipate the influence of their bids and their opponent's bids on the price, their interaction can be modelled as a game.

$$\boxed{\begin{array}{ll} J_i(w^{-i}) & \underset{w_i}{\text{maximize}} \quad U_i \left(\frac{w_i}{\sum_{j \in \mathcal{N}} w_j} C \right) - w_i \\ & \text{subject to} \quad w_i \geq 0. \end{array}}$$

Let $w^* = (w_1^*, \dots, w_N^*)$ be the (unique) Nash equilibrium of this game and let $x^* = w^* \frac{C}{\sum_{j \in \mathcal{N}} w_j^*}$. The worst case efficiency of the proportional allocation mechanism is the ratio

$$\rho = \inf_{\Phi \in \mathcal{F}} \frac{\theta(x^*)}{\max_{z \in \mathbb{C}} \theta(z)}.$$

It was shown in [Joh04] that this ratio is $3/4$.

Observe that the use of a mechanism effectively alters the game. The strategies of the players are now their bids, the feasibility of the allocation, which in a competitive equilibrium is determined by the seller is now in the hands of the administrator who aggregates these bids. Furthermore, the efficiency claimed holds only for the particular mechanism used.

In comparison, we directly consider the generalized Nash game over the space of allocations. As shown in Section 5.3.1, the VE for a game with these objective functions has (worst case) efficiency one. \square

■ 5.4 A general efficiency bound

Most work on efficiency of game theoretic solutions concentrates on the worst case efficiency over a class of objective functions. This section falls into that category. We consider the class of objective functions \mathcal{F} and derive the worst case efficiencies of the solution concepts of the GNE and the VE.

In the case of the GNE as well as the VE, we show that the worst case efficiency is achieved for the class of linear objective functions \mathcal{L} . The worst case efficiency is then calculated constructively. This part of the analysis that uses estimation with linear objective functions follows lines similar

5.4 A general efficiency bound

to those in [JT04]. In particular, we require the following lemma, which is similar to the one used in [JT04].

Lemma 5.1 *Suppose Assumption 5.1 holds. Let $\Phi \in \mathcal{F}$ and let x^* a solution of (SYS). For every $x \in \mathbb{C}$, we have*

$$\frac{\theta(x)}{\theta(x^*)} \geq \frac{\nabla\theta(x)^T x}{\nabla\theta(x)^T x^\ell}, \quad (5.3)$$

where x^ℓ solves the problem (SYS) linearized at x , $(\text{SYS}^\ell(x))$:

$\begin{aligned} \text{SYS}^\ell(x) \quad & \underset{z}{\text{maximize}} \quad \nabla\theta(x)^T z \\ & \text{subject to} \quad z \in \mathbb{C}. \end{aligned}$

Proof : Since θ is concave, the following inequality holds:

$$\theta(\bar{x}) \leq \theta(x) + \nabla\theta(x)^T (\bar{x} - x). \quad (5.4)$$

Now consider the ratio $\frac{\theta(x)}{\theta(x^*)}$. Adding and subtracting $\nabla\theta(x)^T x$ in the numerator and using (5.4) with $\bar{x} = x^*$, it follows that this ratio satisfies

$$\frac{\theta(x)}{\theta(x^*)} \geq \frac{[\theta(x) - \nabla\theta(x)^T x] + \nabla\theta(x)^T x}{[\theta(x) - \nabla\theta(x)^T x] + \nabla\theta(x)^T x^*}. \quad (5.5)$$

By definition of x^ℓ , $\nabla\theta(x)^T x^\ell \geq \nabla\theta(x)^T x^*$, and since both these terms are positive, using this inequality in (5.5) gives

$$\frac{\theta(x)}{\theta(x^*)} \geq \frac{[\theta(x) - \nabla\theta(x)^T x] + \nabla\theta(x)^T x}{[\theta(x) - \nabla\theta(x)^T x] + \nabla\theta(x)^T x^\ell} \quad (5.6)$$

Now, from (5.4), taking $\bar{x} = 0$, we get

$$\theta(x) - \nabla\theta(x)^T x \geq \theta(0), \quad (5.7)$$

which is nonnegative, by Assumption 5.1. Furthermore Assumption 5.1 also provides that $\nabla\theta(x)$ is nonnegative whereby $\nabla\theta(x)^T x$ and $\nabla\theta(x)^T x^\ell$ are both nonnegative. So therefore, dropping the nonnegative term ' $\theta(x) - \nabla\theta(x)^T x$ ' from the numerator and denominator of (5.6) and recalling that

$\nabla\theta(x)^T x^\ell \geq \nabla\theta(x)^T x$, we obtain,

$$\frac{\theta(x)}{\theta(x^*)} \geq \frac{\nabla\theta(x)^T x}{\nabla\theta(x)^T x^\ell},$$

which is the desired result. ■

■ 5.4.1 Worst case efficiency of the VE

Suppose we allow Φ to be any function in \mathcal{F} , i.e., $\varphi_1, \dots, \varphi_N$ are utility functions satisfying Assumption 5.1. In this section we ask the following question: what is the worst possible efficiency that the GNE and the VE can yield when Φ varies over the class \mathcal{F} ? In other words, what are the ratios $\underline{\vartheta}$ and $\underline{\rho}$? We prove that the worst case efficiency of the VE over is in fact zero, i.e. $\underline{\vartheta} = 0$. It follows that $\underline{\rho} = 0$.

Our argument proceeds by characterizing $\underline{\vartheta}$. Theorem 5.3 shows that the worst case efficiency of the VE over \mathcal{F} in fact the same as the worst case efficiency over \mathcal{L} . To prove this, let us set up some notation. For arbitrary $\bar{x} \in \mathbb{C}$ and $\Phi \in \mathcal{F}$, define the linearized functions $\tilde{\varphi}_i^{\bar{x}}, i \in \mathcal{N}$ and $\tilde{\theta}^{\bar{x}}$ as

$$\tilde{\varphi}_i^{\bar{x}}(x) = \nabla\varphi_i(\bar{x})^T x, \quad i \in \mathcal{N} \quad \text{and} \quad \tilde{\theta}^{\bar{x}}(x) = \nabla\theta(\bar{x})^T x = \sum_{j \in \mathcal{N}} \tilde{\varphi}_j^{\bar{x}}(x).$$

Clearly the utility functions $\tilde{\Phi}^{\bar{x}} := (\tilde{\varphi}_1^{\bar{x}}, \dots, \tilde{\varphi}_N^{\bar{x}})$ lie in the class \mathcal{L} . Let $\tilde{F}^{\bar{x}}$ denote the mapping

$$\tilde{F}^{\bar{x}}(x) = - \begin{pmatrix} \nabla_1 \tilde{\varphi}_1^{\bar{x}}(x) \\ \vdots \\ \nabla_N \tilde{\varphi}_N^{\bar{x}}(x) \end{pmatrix}.$$

Observe that

$$\tilde{F}^{\bar{x}}(x) = - \begin{pmatrix} \nabla_1 \varphi_1(\bar{x}) \\ \vdots \\ \nabla_N \varphi_N(\bar{x}) \end{pmatrix} = F(\bar{x}) \quad \forall x.$$

This also shows that every VE of a game with $\Phi \in \mathcal{F}$ is a VE of some game with linear objectives.

Lemma 5.2 *Let $\Phi \in \mathcal{F}$. Then*

$$\text{VE}(\Phi) \subseteq \bigcup_{x \in \text{VE}(\Phi)} \text{VE}(\tilde{\Phi}^x).$$

5.4 A general efficiency bound

Proof : Since \bar{x} is a VE for objective functions Φ ,

$$F(\bar{x})^T(y - \bar{x}) \geq 0 \quad \forall y \in \mathbb{C} \implies \tilde{F}^{\bar{x}}(\bar{x})^T(y - \bar{x}) \geq 0 \quad \forall y \in \mathbb{C}.$$

This in turn implies $\bar{x} \in \text{VE}(\tilde{\Phi}^{\bar{x}})$. ■

Following the central result of this section. We show that for the worst case efficiency of the VE over the class \mathcal{F} , \mathcal{V} , it suffices to look at the class of linear objective functions, \mathcal{L} .

Theorem 5.3 *Suppose Assumption 5.1 holds. Then*

$$\underline{\vartheta} = \inf_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} = \inf_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{L}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)}.$$

Proof : Lemma 5.1 shows that for $\Phi \in \mathcal{F}$ and $\bar{x} \in \text{VE}(\Phi)$,

$$\frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \geq \frac{\tilde{\theta}^{\bar{x}}(\bar{x})}{\max_{z \in \mathbb{C}} \tilde{\theta}^{\bar{x}}(z)}.$$

So for any $\Phi \in \mathcal{F}$,

$$\begin{aligned} \inf_{\bar{x} \in \text{VE}(\Phi)} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} &\geq \inf_{\bar{x} \in \text{VE}(\Phi)} \frac{\tilde{\theta}^{\bar{x}}(\bar{x})}{\max_{z \in \mathbb{C}} \tilde{\theta}^{\bar{x}}(z)} \\ &\geq \inf_{\bar{x} \in \text{VE}(\tilde{\Phi}^{\bar{x}}), \bar{x} \in \text{VE}(\Phi)} \frac{\tilde{\theta}^{\bar{x}}(\bar{x})}{\max_{z \in \mathbb{C}} \tilde{\theta}^{\bar{x}}(z)}, \quad \text{by Lemma 5.2,} \\ &= \inf_{\bar{x} \in \text{VE}(\tilde{\Phi}^{\bar{x}}), \bar{x} \in \text{VE}(\Phi)} \frac{\sum_{i \in \mathcal{N}} \tilde{\varphi}_i^{\bar{x}}(\bar{x})}{\max_{z \in \mathbb{C}} \sum_{i \in \mathcal{N}} \tilde{\varphi}_i^{\bar{x}}(z)}, \\ &\geq \min_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{L}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)}, \quad \text{since } \{\tilde{\Phi}^{\bar{x}} : \bar{x} \in \text{VE}(\Phi)\} \subseteq \mathcal{L}. \end{aligned}$$

Since this holds for any $\Phi \in \mathcal{F}$, we must have

$$\inf_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \geq \inf_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{L}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)}.$$

But since $\mathcal{L} \subseteq \mathcal{F}$, we must also have

$$\inf_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{F}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \leq \inf_{\bar{x} \in \text{VE}(\Phi), \Phi \in \mathcal{L}} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)}.$$

Chapter 5. The Efficiency of Generalized Nash Equilibria

The result follows. ■

The above result does not provide us the value of $\underline{\vartheta}$ but only a characterization of it. We now show that $\underline{\vartheta} = 0$ through an example.

Example 5.2. Consider a game with objective functions $\varphi_i(x) = d_i^T x$, where $d_i = (d_i^1, \dots, d_i^N) \in \mathbb{R}^N$ such that $\nabla \theta = \sum d_i \geq 0$. Furthermore, assume that $d_i^i > 0$ for all $i \in \mathcal{N}$. It is easy to see this collection of objective functions Φ satisfies Assumption 5.1.

Let $c_i = d_i^i$ and $c := (c_1, \dots, c_N)^T = F(x)$, for each x . The set of VEs of this game, $\text{VE}(c)$ is the set of x for which there exists λ such that

$$0 \leq x \perp -c + \lambda \mathbf{1} \geq 0$$

$$0 \leq \lambda \perp C - \mathbf{1}^T x \geq 0$$

Since $c > 0$, observe that any λ satisfying these equilibrium conditions must be strictly positive and that $\mathbf{1}^T x = C$ must hold for any VE x .

Now consider the following values for d_1, \dots, d_N . Let $\varepsilon \in (0, 1)$ and let

$$d_i^j := \begin{cases} 0, & \forall j \neq i, \\ \varepsilon, & j = i. \end{cases} \quad \forall i \in \mathcal{N} \setminus \{1, N\}, \quad d_1^j := \begin{cases} \frac{1}{N-1} & \forall j \neq 1, \\ \varepsilon & j = 1, \end{cases} \quad d_N^j := \begin{cases} 0 & \forall j \neq N, \\ 2\varepsilon & j = N. \end{cases}$$

It follows that $c = (\varepsilon, \dots, \varepsilon, 2\varepsilon)$ and $d = \sum d_i = (\varepsilon + 1, \varepsilon, \dots, \varepsilon, 2\varepsilon)$. It is easy to check that $x^* := (0, \dots, 0, C)$ is a VE (the equilibrium conditions are satisfied for $\lambda = 2\varepsilon$). Since $\varepsilon + 1 > 2\varepsilon$, the optimal value of (SYS) is $C(\varepsilon + 1)$. The worst case efficiency of the VE is bounded above by the efficiency of x^* . Therefore,

$$\underline{\vartheta} \leq \frac{c^T x^*}{\max_{z \in \mathcal{C}} d^T z} = \frac{2C\varepsilon}{C(\varepsilon + 1)} = \frac{2\varepsilon}{\varepsilon + 1}.$$

Letting ε decrease to zero reveals that the worst case efficiency $\underline{\vartheta} = 0$, implying that the VE can be arbitrarily inefficient. Letting ε approach one shows that efficiencies arbitrarily close to unity are also achievable in the class \mathcal{F} . □

Summarizing the above example we have the following result for the worst case efficiency of the

5.5 Games where the VE is efficient

VE.

Theorem 5.4 *The worst case efficiency of the VE (and the GNE) over the class \mathcal{F} is zero. This bound is tight in the sense that for any $\varepsilon \in (0, 1]$, there exists a game with objective functions Φ in \mathcal{F} such that a VE (and hence a GNE) of this game has efficiency ε .*

Proof : We only need to prove the tightness. That efficiency of ε for any $\varepsilon \in (0, 1)$ are achieved is demonstrated in the example above. Section 5.3.1 showed that in the setting of perfect competition, the VE coincides with the competitive equilibrium and is therefore efficient. Thus efficiency $\varepsilon = 1$ is also achievable. ■

■ 5.5 Games where the VE is efficient

In this section we present the *best* case analysis of the efficiency of the GNE and the VE. Although Section 5.3.1 and Theorem 5.4 demonstrate that the best case efficiency of the VE and the GNE is unity, we go a step further in this section and characterize the class of games for which the VE is efficient. i.e. the best *and* the worst case efficiency of the VE is unity. This class is large and it includes in it the perfectly competitive setting. Then, restricting ourselves to this class we determine the worst case efficiency of the GNE over this class. We find that while the VE is efficient for every game in this class, the GNE can be arbitrarily inefficient.

For the VE to be efficient, it has to be optimal for the problem (SYS). To find a sufficient condition for the VE to solve (SYS), let us first answer a related question: is there an optimization problem that the VE solves? By this we seek another optimization problem whose optimality conditions are the same as the equilibrium conditions of the VE, whereby solving the two is equivalent. Fortunately the answer to this is rather simple. The VE is the solution of $\text{VI}(\mathbb{C}, F)$. If there exists a concave function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ such that

$$-\nabla f(x) = F(x) \quad \forall x \in \mathbb{R}^N, \quad (5.8)$$

then $\text{VI}(\mathbb{C}, F)$ is equivalent to the $\text{VI}(\mathbb{C}, -\nabla f)$ which, by the convexity of \mathbb{C} is equivalent to the

optimization problem

VE-OPT	$\begin{aligned} & \underset{x}{\text{maximize}} && f(x) \\ & \text{subject to} && \sum_{j \in \mathcal{N}} x_j \leq C, \\ & && x \geq 0. \end{aligned}$
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One way of interpreting (5.8) is that it asks for F to be integrable and for $-f$ to be its integrand. It is well known that such a function f exists if and only if the Jacobian $\nabla F(x)$ is symmetric [OR87].

Suppose such a function does exist. Then since $-\nabla f(x) = F(x)$ holds for all x , we must have for each i

$$\begin{aligned} \nabla_i [f(x) - \varphi_i(x)] &= 0 \\ \text{i.e., } f(x) - \varphi_i(x) &= \eta_i(x^{-i}), \end{aligned} \tag{5.9}$$

for some η_i which is a function of x^{-i} and for all $i \in \mathcal{N}$. Using this, one may now ask the original question. For what objective functions Φ is $f(x) = \theta(x) = \sum_{j \in \mathcal{N}} \varphi_j(x)$? The following theorem provides us such a characterization.

Theorem 5.5 *Suppose Assumption 5.1 holds. The identity $-\nabla \theta = F$ holds if and only if there exist continuously differentiable functions η_i of x^{-i} for each $i \in \mathcal{N}$, such that the utility functions are given by*

$$\varphi_i(x) = \frac{\sum_{j=1}^N \eta_j(x^{-j})}{N-1} - \eta_i(x^{-i}), \tag{5.10}$$

for every $i \in \mathcal{N}$.

Proof : “ \implies ” Let $-\nabla \theta = F$. In particular, we have, for each $i \in \mathcal{N}$, and for all x , $\nabla_i \theta(x) = \nabla_i \varphi_i(x)$. Recall from Assumption 5.1 that θ is concave. Therefore (5.9) gives that for each $i \in \mathcal{N}$,

$$\theta(x) - \varphi_i(x) = \eta_i(x^{-i}), \tag{5.11}$$

for some η_i which is a function of x^{-i} . Since φ_i and θ are continuously differentiable, it follows that

5.5 Games where the VE is efficient

η_i is continuously differentiable. Summing (5.11) over i and using that $\theta = \sum \varphi_i$, gives

$$\theta(x) = \frac{\sum_{i=1}^N \eta_i(x^{-i})}{N-1}. \quad (5.12)$$

Now substituting for θ from (5.12) in (5.11) gives the result desired in (5.10).

“ \Leftarrow ” Suppose (5.10) holds for some continuously differentiable functions η_1, \dots, η_N . Summing (5.10) over i gives (5.12). Then for each $i \in \mathcal{N}$,

$$\nabla_i \varphi_i(x) = \nabla_i \left(\frac{\sum_{j=1}^N \eta_j(x^{-j})}{N-1} \right) = \nabla_i \theta(x).$$

In other words, $-\nabla \theta = F$. ■

Theorem 5.5 shows that for a class of objective functions, the worst and best case efficiency of the VE is unity.

Theorem 5.6 *Suppose Assumption 5.1 holds. Then the best case efficiencies of the VE and GNE are unity. i.e. $\bar{\rho} = \bar{\vartheta} = 1$.*

Remarkably, the class given by (5.10) does not depend on \mathbb{C} . i.e. if $\varphi_1, \dots, \varphi_N$ are of the form given by (5.10), then the $\text{VI}(V, F)$ is equivalent to $\text{VI}(V, -\nabla \theta)$ for any closed convex set V . Note also that from (5.9),

$$\eta_i(x^{-i}) = \sum_{j \in \mathcal{N}} \varphi_j(x) - \varphi_i(x) = \sum_{j \neq i} \varphi_j(x).$$

In general the right hand side may not independent of x_i , and the above equation may not hold. Thus, not every game has VEs equivalent to optimization problems.

We now revisit the relationship of the VE with the competitive equilibrium in the light of this result.

Example 5.3. In Section 5.3.1, we considered a perfectly competitive setting and then we considered a shared-constraint game analogous to it. The game considered therein was with $\varphi_i(x) = U_i(x_i)$ for all i . For this game, we indeed have the functions η_i given by

$$\eta_i(x^{-i}) = \sum_{j \neq i} U_j(x_j),$$

which is independent of x_i . □

Notice that the statement ‘ $-\nabla\theta = F$ ’ characterized by Theorem 5.5 is somewhat stronger than the statement ‘every VE is efficient’. ‘ $-\nabla\theta = F$ ’ provides the equivalence between the VIs characterizing the VE and solutions of (SYS). This equivalence implies the equality of the solution sets of the said VIs, though it is not necessary to conclude the equality of these sets. Following is an example of a game where objectives do not satisfy (5.10), every VE is efficient.

Example 5.4. Consider $\varphi_i(x) = x_i g(\mathbf{1}^T x)$ and assume that $g(\mathbf{1}^T x) + x_i g'(\mathbf{1}^T x) > 0$ and $\nabla\theta(x) = g(\mathbf{1}^T x) + (\mathbf{1}^T x)g'(\mathbf{1}^T x) \geq 0$ for all x . It is easy to check that this set of functions does not satisfy (5.10). But, as we show below, every VE of this game is efficient.

Let us first consider the system problem (SYS). Since $\nabla\theta \geq 0$, the maximum in (SYS) is attained at $\mathbf{1}^T x = C$, whereby the optimal value of (SYS) is $Cg(C)$. Now consider the shared-constraint game formed from these utilities. Thus x is a VE of this game if and only if

$$\begin{aligned} 0 &\leq x \perp -g(\mathbf{1}^T x)\mathbf{1} - xg'(\mathbf{1}^T x) + \lambda\mathbf{1} \geq 0 \\ 0 &\leq \lambda \perp C - \mathbf{1}^T x \geq 0, \end{aligned}$$

for some $\lambda \in \mathbb{R}$. Since $g(\mathbf{1}^T x)\mathbf{1} + xg'(\mathbf{1}^T x) > 0$, $\lambda = 0$ does not satisfy these equations. Consequently, for each VE x , the equality $\mathbf{1}^T x = C$ must hold. So for any VE x , $\theta(x) = \mathbf{1}^T x g(\mathbf{1}^T x) = Cg(C)$, which is the optimal value of (SYS). □

Also note that the requirement ‘ $-\nabla\theta = F$ ’ is, strictly speaking, not the same as ‘ $\text{VI}(\mathbb{C}, F) \equiv \text{VI}(\mathbb{C}, -\nabla\theta)$ ’, because the solution set of a VI is invariant under multiplication of the function by a positive constant. Specifically, if $-c\nabla\theta = F$, where $c > 0$ is a real number, then $\text{VI}(\mathbb{C}, F)$ is equivalent to $\text{VI}(\mathbb{C}, -\nabla\theta)$. Therefore if $F = -\nabla f$ and $f = c\theta$, we get using (5.9),

$$c\theta(x) = \varphi_i(x) + \eta_i(x^{-i}).$$

Arguing as in Theorem 5.5, we get

$$\theta(x) = \frac{\sum_{j \in \mathcal{N}} \eta_j(x^{-j})}{cN - 1} \quad \text{and} \quad \varphi_i(x) = \frac{\sum_{j \in \mathcal{N}} \eta_j(x^{-j})}{cN - 1} - \eta_i(x^{-i}) \quad \forall x.$$

5.5 Games where the VE is efficient

Finally, we note that (5.9) has also appeared previously in literature. Slade in [Sla94] has derived (5.9) as a means of giving sufficient conditions for the stationarity conditions of an unconstrained Nash equilibrium to be equivalent to an optimization problem.

We invoke this class of games in the following section. For reference, denote

$$\begin{aligned}\mathcal{F}' &= \{\Phi \mid \Phi \in \mathcal{F} \text{ and } \exists \text{ functions } \eta_i \text{ of } x^{-i}, i \in \mathcal{N}, \text{ so that } \varphi_1, \dots, \varphi_N \text{ are given by (5.10)}\} \\ \mathcal{L}' &= \{\Phi \mid \Phi \in \mathcal{L} \text{ and } \exists \text{ functions } \eta_i \text{ of } x^{-i}, i \in \mathcal{N}, \text{ so that } \varphi_1, \dots, \varphi_N \text{ are given by (5.10)}\}.\end{aligned}\tag{5.13}$$

Denote the efficiencies of the GNE over the class \mathcal{F}' by

$$\bar{\rho}' := \sup_{x \in \text{GNE}(\Phi), \Phi \in \mathcal{F}'} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \quad \text{and} \quad \underline{\rho}' := \inf_{x \in \text{GNE}(\Phi), \Phi \in \mathcal{F}'} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \tag{5.14}$$

Clearly, $\bar{\rho}' = 1$ and $\underline{\rho}' \geq \underline{\rho}$. The best and worst case efficiencies of the VE,

$$\bar{\vartheta}' := \sup_{x \in \text{VE}(\Phi), \Phi \in \mathcal{F}'} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \quad \text{and} \quad \underline{\vartheta}' := \inf_{x \in \text{VE}(\Phi), \Phi \in \mathcal{F}'} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)}$$

are both unity.

In the following section the worst case efficiency of the GNE to the class of functions \mathcal{F}' is addressed. i.e. we calculate ratio $\underline{\rho}'$. We find that this ratio is in fact zero, indicating that the GNE can be arbitrarily inefficient even while the VE is efficient.

■ 5.5.1 Worst case efficiency of the GNE

Recall the linearized objective functions from Section 5.4.1 $\tilde{\Phi}^{\bar{x}}$ where $\bar{x} \in \mathbb{C}$ is the point of linearization. We first show that a result similar to Theorem 5.3 holds, thanks to which $\underline{\rho}'$ is the worst case efficiency of GNE over the class \mathcal{L}' .

Theorem 5.7 *Suppose Assumption 5.1 holds. Then the ratio $\underline{\rho}'$ defined in (5.14) is given by*

$$\underline{\rho}' = \inf_{x \in \text{GNE}(\Phi), \Phi \in \mathcal{L}'} \frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)}.$$

Chapter 5. The Efficiency of Generalized Nash Equilibria

Proof : The proof is similar to that of Theorem 5.3, so we will only sketch it. By repeating the arguments in Lemma 5.2, one can conclude that for any $\Phi \in \mathcal{F}'$,

$$\text{GNE}(\Phi) \subseteq \bigcup_{x \in \text{GNE}(\Phi)} \text{GNE}(\tilde{\Phi}^x), \quad (5.15)$$

where the notation $\tilde{\Phi}^x$ stands for the linearized version of Φ , as is Section 5.4.1. By Lemma 5.1, for any $\Phi \in \mathcal{F}'$ and $\bar{x} \in \text{GNE}(\Phi)$,

$$\frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \geq \frac{\tilde{\theta}^{\bar{x}}(\bar{x})}{\max_{z \in \mathbb{C}} \tilde{\theta}^{\bar{x}}(z)}.$$

We see that, if Φ belongs to the class \mathcal{F}' , i.e. if $F(x) = -\nabla \theta(x)$ for all x , then the linearization $\tilde{\Phi}^{\bar{x}}$ belongs to \mathcal{L}' . i.e. we have $\tilde{F}^{\bar{x}}(x) = -\nabla \tilde{\theta}^{\bar{x}}(x)$ for all x . Similar to the proof of Theorem 5.3, using (5.15), we get that for each $\Phi \in \mathcal{F}$,

$$\inf_{\bar{x} \in \text{GNE}(\Phi)} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)} \geq \inf_{\bar{x} \in \text{GNE}(\Phi), \Phi \in \mathcal{L}'} \frac{\theta(\bar{x})}{\max_{z \in \mathbb{C}} \theta(z)}.$$

Then using that $\mathcal{L}' \subseteq \mathcal{F}$ completes the proof. ■

Once again we obtain that the linear case is where the worst case of efficiency is achieved. We now show through an example that this efficiency $\underline{\rho}'$ is indeed zero.

Example 5.5. Consider a game with objective functions $\varphi_i(x) = d_i^T x$ for each $i \in \mathcal{N}$, where $d_i = (d_i^1, \dots, d_i^N) \in \mathbb{R}^N$ such that $d := \nabla \theta = \sum d_i \geq 0$ and $d_i^i > 0$ for all $i \in \mathcal{N}$. It is easy to see this collection of objective function Φ satisfies Assumption 5.1. Let $c_i = d_i^i$ and

$$c := \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} \nabla_1 \varphi_1(x) \\ \vdots \\ \nabla_N \varphi_N(x) \end{pmatrix} = F(x), \quad \forall x.$$

Furthermore we assume

$$\nabla \theta = \sum d_i = c.$$

5.5 Games where the VE is efficient

Let $\text{GNE}(c)$ denote the set of GNEs of this game. Recall that x is a GNE if and only if there exists $\Lambda = (\lambda_1, \dots, \lambda_N)^T$ such that

$$\begin{aligned} 0 &\leq x \perp -c + \Lambda \geq 0, \\ 0 &\leq \Lambda \perp \mathbf{1}(C - \mathbf{1}^T x) \geq 0. \end{aligned}$$

Observe that any Λ satisfying these equilibrium conditions must be strictly positive in all components because c is assumed to be so. This implies that $\mathbf{1}^T x = C$ must hold for any GNE x . In fact, $\mathbf{1}^T x = C$ along with $x \geq 0$ is also sufficient for x to be a GNE; the equilibrium conditions are then satisfied by $\Lambda = c$. So we get

$$\text{GNE}(c) \subseteq \{x \mid \mathbf{1}^T x = C, x \geq 0\}.$$

Now let $1 > \varepsilon > 0$ and take $c = c_* := (1, \varepsilon, \varepsilon, \dots, \varepsilon)$. The point $x^* := (0, \dots, 0, C)$ is a GNE for this c . The worst case efficiency of the GNE no greater than the efficiency of the GNE x^* for the game with $c = c_*$. Therefore,

$$\underline{\rho}' \leq \frac{c_*^T x^*}{\max_{z \in \mathbb{C}} c_*^T z} = \frac{C\varepsilon}{C} = \varepsilon.$$

Evidently, since ε may take any value in $(0, 1)$, the worst case efficiency $\underline{\rho}' = 0$. In other words, the GNE can be arbitrarily inefficient. \square

Combining the above example with the best case efficiency of the GNE (i.e. $\varepsilon = 1$), we get our final result.

Theorem 5.8 *Suppose Assumption 5.1 holds. The worst case efficiencies of the GNE over the class \mathcal{F} and over class \mathcal{F}' are both zero. This bound is tight, in the sense that for every $\varepsilon \in (0, 1]$ there is a game satisfying Assumption 5.1 for which every VE is efficient but has a GNE that has efficiency ε .*

This is a particularly surprising result for it clearly shows the disparity between the VE and GNE. Example 5.5 also indicates the cause for loss of efficiency in the GNE. Games like these often admit a manifold of GNEs. In Example 5.5, the set $\{x \mid x \geq 0, \mathbf{1}^T x = C\}$ is this manifold. When the utility functions are linear, it is always possible to find a GNE that allocates zero quantity the

player with the highest contribution to θ (i.e. the player i with the largest $\nabla_i \varphi_i$). One may also conclude that this, somewhat pathological property of the GNE, makes it unattractive as a solution concept for shared-constraint resource allocation games.

■ 5.6 Remediating zero worst case efficiency

The possibility of arbitrarily low efficiency in the case of the GNE and the VE is indeed disappointing for it implies that both these solution concepts can yield equilibria that are as inefficient as possible. In this section we discuss some ways in this could possibly be remedied.

■ 5.6.1 Utility functions with bounded gradients

Looking back at Example 5.5 we see that the limiting efficiency of zero is achieved as the gradient of θ approaches the zero-vector. In the case of linear utilities, this corresponds to the possibility of constant zero utility. This is perhaps a pathological situation that does not occur in realistic settings. This observation also suggests a way for salvaging nonzero efficiency. Consider the following class of functions for $0 < \alpha < \beta < \infty$,

$$\mathcal{F}^{[\alpha, \beta]} = \{\Phi \mid \Phi \in \mathcal{F} \text{ and } \beta \mathbf{1} \geq \nabla \theta(x) \geq \alpha \mathbf{1} \quad \forall x \in \mathbb{C}\}.$$

Utility functions Φ belonging to this class have the gradients of their sum bounded above and below. While for a given continuously differentiable function θ , a finite bound on the gradient over the compact domain \mathbb{C} always exists. $\mathcal{F}^{[\alpha, \beta]}$ is a collection of functions, gradients of all of which are bounded by the same bound.

Let $\Phi \in \mathcal{F}$. Recall Lemma 5.1, which showed that for any $x \in \mathbb{C}$, the ratio $\frac{\theta(x)}{\theta(x^*)}$, where $x^* \in \text{SOL}(\text{SYS})$ is was bounded below by the ratio $\frac{\nabla \theta(x)^T x}{\nabla \theta(x^*)^T x^*}$ where x^* solves $(\text{SYS}^\ell(x))$. Now if $\Phi \in \mathcal{F}^{[\alpha, \beta]}$, then for any $x \in \mathbb{C}$,

$$\frac{\theta(x)}{\theta(x^*)} \geq \frac{\alpha \mathbf{1}^T x}{\beta \mathbf{1}^T x^*}.$$

Furthermore since Assumption 5.1 ensures that $F(x) > 0$ whereby, for any GNE x , $\mathbf{1}^T x = C$. Likewise, for the problem $(\text{SYS}^\ell(x))$ optimality of x^ℓ implies $\mathbf{1}^T x^\ell = C$. This immediately gives us

5.6 Remedying zero worst case efficiency

that

$$\inf_{x \in \text{GNE}(\Phi)} \frac{\theta(x)}{\theta(x^*)} \geq \frac{\alpha}{\beta}.$$

Since this holds for all $\Phi \in \mathcal{F}^{[\alpha, \beta]}$, the worst case efficiency of the GNE over this class is $\frac{\alpha}{\beta}$. It follows that the worst case efficiency of the VE over the class $\mathcal{F}^{[\alpha, \beta]}$ is greater than or equal to $\frac{\alpha}{\beta}$.

Example 5.6. Consider the following class of *exponential* utility functions.

$$\varphi_i(x) = 1 - \exp(-d_i^T x), \quad \forall i \in \mathcal{N},$$

where for each i , the vector of coefficients $d_i = (d_i^1, \dots, d_i^N)$ is chosen from a compact set $D \subseteq \mathbb{R}_+^N$ and with $d_i^i > 0$. It follows that these utility functions satisfy Assumption 5.1. It is easy to see that the following bounds hold.

$$\alpha \sum d_i \leq \nabla \theta(x) = \sum_{i \in \mathcal{N}} d_i \exp(-d_i^T x) \leq \sum d_i, \quad \forall x \in \mathbb{C},$$

where α is at least the value α^0 given by

$$\alpha^0 = \min_{(d, z) \in D \times \mathbb{C}} \exp(-d^T z).$$

α^0 is positive because D is compact. It follows that $\frac{\theta(x)}{\theta(x^*)} \geq \alpha$, and that the worst case efficiency of the GNE for this class of games is at least α . \square

■ 5.6.2 Other notions of efficiency

The notion of what constitutes efficient allocations varies with the setting in question and need not always correspond to the allocation that maximizes the sum of the objective functions of players. This is particularly true when what players are maximizing are not merely their utilities but some modifications of them. The system-level goal though, from the point of view of social welfare, remains the maximization of aggregate utility. As mentioned in Section 5.3.2, mechanism design employs such a paradigm. The optimization problems that players solve are not of ‘utility maximization’ but instead of ‘*payoff maximization*’, though the efficiency is benchmarked on the

aggregate utility. And under this interpretation, greater efficiency may be obtained.

In this section, we study a particular kind of model that uses the above philosophy and provide a general principle for the VE to be efficient in this new sense. This model is adapted from and is a somewhat simplified version of the model in [AB02]. We first explain the model from [AB02]. Similar models are used in [KS03, LA00, MW00].

Example 5.7. The game consists of N players that attempt to access bandwidth over a general network with a set of links L . The subset of links used by player i are L_i and the utility received by player i is $U_i(x_i)$ for flow x_i that it obtains. If x is the vector of typical flows of all players, the flows permitted by the network are those in the set $\{x \mid x \geq 0, \mathbf{A}x \leq \mathbf{C}\}$ where \mathbf{A} is a $|L| \times N$ matrix whose element $\mathbf{A}[\ell, i]$ is 1 if player i uses link ℓ and \mathbf{C} is the $|L|$ -vector of capacities the links. The network is however subject to delay, the cost of which is seen by a player for every link it uses. Specifically, each player faces a cost which is the sum of the costs on the links that it uses,

$$c^i(x) = \sum_{\ell \in L_i} c_\ell \left(\sum_{j: \ell \in L_j} x_j \right).$$

Each player is thus faced with the following optimization problem.

$\text{AB}_i(x^{-i})$	$\begin{aligned} & \underset{x_i}{\text{maximize}} && \varphi_i(x) := U_i(x_i) - c^i(x) \\ & \text{subject to} && \mathbf{A}x \leq \mathbf{C}, \\ & && x_i \geq 0. \end{aligned}$
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The game formed from these problems is clearly a shared-constraint game, more general than the game we have considered. Our game can be thought of a special case of this game with one link, i.e. $|L| = 1$.

The system-level problem defined⁷ in [AB02] however is *not* to maximize the sum of the objectives $\sum \varphi_i$, but instead to maximize the aggregate utility less the total delay over the network.

⁷The constraint ' $\mathbf{A}x \leq \mathbf{C}$ ' is not used in the system problem in [AB02]. This constraint is irrelevant in their case since the costs approach infinity as this constraint becomes active.

5.6 Remedying zero worst case efficiency

Specifically, an *efficient allocation* in this case is the solution of following problem.

SYS-AB	$\begin{aligned} & \underset{x}{\text{maximize}} && \sum_{i \in \mathcal{N}} U_i(x_i) - \sum_{\ell \in L} c_\ell \left(\sum_{j: \ell \in L_j} x_j \right) \\ & \text{subject to} && \mathbf{A}x \leq \mathbf{C}, \\ & && x \geq 0. \end{aligned}$
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Notice that while the delay on a particular link ℓ appears in the optimization problem of possibly multiple players, it is accounted for only once in (SYS-AB). The motivation for this is that, while each player that experiences the delay on ℓ suffers a cost c_ℓ due to it, the *system* goal is to merely minimize aggregate delay in addition to maximizing aggregate utility. In particular, in (SYS-AB) the delay on link ℓ is not scaled by the number of players using it. Under the assumption that c_ℓ approaches infinity as the link ℓ gets congested, it is proved in [AB02] that the GNE of this game is efficient. The GNE and the VE coincide in this case because they are in the interior of the feasible region. \square

We derive a general principle within our setting for a result such as that in [AB02] to hold. We stick to the solution concept of the VE and show that for utility functions taken from the class \mathcal{F}' , the VE of the shared-constraint game is optimal for the system problem analogous to the one above.

Consider the game where player i derives utility $U_i(x)$ from an allocation x but he suffers a cost $\tau(x)$, which we assume is a convex function of x . Player i 's objective then is to solve

$A_i(x^{-i})$	$\begin{aligned} & \underset{x_i}{\text{maximize}} && U_i(x) - \tau(x) \\ & \text{subject to} && \sum_{j \in \mathcal{N}} x_j \leq C, \\ & && x \geq 0. \end{aligned}$
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(SYS) problem however is changed to the following.

SYS	$\begin{aligned} & \underset{x}{\text{maximize}} && \sum_{j \in \mathcal{N}} U_j(x) - \tau(x) \\ & \text{subject to} && \sum_{j \in \mathcal{N}} x_j \leq C, \\ & && x \geq 0. \end{aligned}$
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Let \mathbf{U} be the tuple (U_1, \dots, U_N) and let F be as before,

$$F = - \begin{pmatrix} \nabla_1 U_1 - \nabla_1 \tau \\ \vdots \\ \nabla_N U_N - \nabla_N \tau \end{pmatrix}.$$

We now ask the question: for what utility functions \mathbf{U} does the VE of the shared-constraint game formed from $(A_1), \dots, (A_N)$ solve (SYS)? Similar to the argument in Section 5.5, suppose there exists a concave continuously differentiable function f such that $-\nabla f = F$. It follows that there exist for each i continuously differentiable functions η_i of x^{-i} such that

$$f(x) = U_i(x) - \tau(x) + \eta_i(x^{-i}) \quad \forall x.$$

For the VE to solve (SYS), we need that $f = \theta$, where θ now is

$$\theta = \sum_{j \in \mathcal{N}} U_j - \tau.$$

θ is a concave function if \mathbf{U} satisfies Assumption 5.1. Substituting $f = \theta$ above gives

$$\sum_{j \in \mathcal{N}} U_j(x) = U_i(x) + \eta(x^{-i}), \quad \forall x,$$

for all i . The above equation is the same as (5.11), but for functions \mathbf{U} . It follows that for each i , U_i must be of the form given by (5.10), and in particular in class \mathcal{F}' . This is summarized in the following result.

5.6 Remedying zero worst case efficiency

Theorem 5.9 *Suppose \mathbf{U} satisfies Assumption 5.1 and consider the game formed from $(A_1), \dots, (A_N)$ described above. The identity $-\nabla\theta = F$ holds if and only if \mathbf{U} belongs to the class \mathcal{F}' .*

It is easy to see that the game of Example 5.7 follows as a special of this theorem.

■ 5.6.3 Reserve price

So far we have assumed that there is no administrative intervention in the allocation of the resources. A consequence of this is the possibility that, at equilibrium, “non-serious” players with low marginal utility get large portions of the resource. An apt instance of this situation is the game in Example 5.5, where at equilibrium, the player with the highest marginal utility amongst all players gets no portion of the resource while the player with lowest marginal utility gets all of it. The system problem, on the other hand, allocates all the resource to player with highest marginal utility. Since the lowest marginal utility could be arbitrarily small, the worst case efficiency is zero.

One way to mitigate this problem is to *screen* players so that, at equilibrium, only those players play the game who are “sufficiently serious” about wanting the resource. We show here that this can be done with some, though minimal administrative intervention. We show that, under certain assumptions, efficiencies as high as unity are achievable. Underlying this approach, is the concept of a *reserve price*.

The idea of a reserve price or reservation price can be traced to Myerson’s seminal work on optimal auction design for a single indivisible object in a Bayesian setting [Mye81]. In this setting, a reserve price is set by the auctioneer or seller and revealed to all players with the rider that only those bids will be considered in the auction that are at least large as the reserve price. As a consequence, players whose valuations do not exceed the reserve price are eliminated from the auction. The seller risks keeping the object to himself in the event that the all players bid below this price but he also has a chance for higher revenue⁸.

In the context of allocation of divisible resources that we are concerned with, the reserve price is typically implemented through the pricing formula or mechanism employed, as has been done, e.g., by Maheswaran and Başar [MB03, MB06]. In our setting of a shared-constraint game we do

⁸See [Eng87, NRTV07] for an analysis of the revenue-optimal reserve price in auctions.

Chapter 5. The Efficiency of Generalized Nash Equilibria

not have a pricing formula or a direct handle on the price. We impose the price through a cost or a toll. The administrator fixes a value π so that each player i that receives quantity x_i is charged a cost πx_i . This cost may be monetary and collected by an administrator or it may be a cost induced by a physical inconvenience such as delay which does not require imposition by an administrator. We do not concern ourselves with the manner of imposition of this cost, but only note that even in the case where an administrator imposes it, it can be imposed with minimal intervention. As a result of the reserve price, player i now solves the following optimization problem.

$$\begin{array}{ll} A_i(x^{-i}) & \text{maximize}_{x_i} \quad \varphi_i(x) - \pi x_i \\ & \text{subject to} \quad \sum_{j \in \mathcal{N}} x_j \leq C, \\ & \quad \quad \quad x_i \geq 0. \end{array}$$

We benchmark efficiency with respect to the original system problem.

$$\begin{array}{ll} \text{SYS} & \text{maximize}_x \quad \sum_{j \in \mathcal{N}} \varphi_j(x) \\ & \text{subject to} \quad \sum_{j \in \mathcal{N}} x_j \leq C, \\ & \quad \quad \quad x \geq 0. \end{array}$$

Consider utility functions $\Phi \in \mathcal{F}'$. The argument in Section 5.5.1 shows, it suffices to consider utility functions $\Phi \in \mathcal{L}'$ to evaluate the worst case efficiency, so we restrict ourselves to linear utility functions. In particular, consider for each i , $\varphi_i(x) = d_i^T x$ for all x and $c_i = d_i^i > 0$. Notice that for any i , coefficients d_i^j , with $j \neq i$ do not affect optimal choice of player i and the resulting equilibrium. Furthermore since $\theta(x) = c^T x$, where $c = (c_1, \dots, c_N)$, they do not affect the system problem. So without loss of generality we may take $d_i^j = 0$ for all $i \neq j$. We are thus considering the case of ‘perfectly competitive’ (linear) utility functions.

If the reserve price is larger than the marginal utility of a player \hat{i} , i.e. $\pi > c_{\hat{i}}$, then it is optimal for \hat{i} to ask for zero quantity. For the other players, the shared constraint is now the set

$$\left\{ x^{-\hat{i}} \mid x^{-\hat{i}} \geq 0, \sum_{j \neq \hat{i}} x_j \leq C \right\}.$$

5.6 Remedying zero worst case efficiency

Effectively, player \hat{i} is eliminated and we are left with a game similar to the original game, but amongst players $\mathcal{N} \setminus \hat{i}$. The imposition of a reserve price filters players with “low interest” in the resource and retains only those players who gain utility at least π from a unit of the resource.

There is, of course, a possibility of *overdoing* this elimination by eliminating *all* players in the competition. Indeed, for a given c , one can always find a price $\pi = \pi^*$, where $\pi^* > \max_i c_i$ so that for each player, it is optimal to demand zero, and no resource is allocated. i.e. the allocation $(0, \dots, 0)$ is the (unique) equilibrium. This is akin to the nonzero probability in the Bayesian single object auction, that the seller has to keep the item for itself with zero revenue. In that setting, one can counter possibility by arguing that the *expected* profit of the seller is positive for a certain price [NRTV07]. To rule out this possibility in our setting, we are compelled to assume that there is at least one player who is not eliminated. With this assumption, the imposition of a reserve price leads to improvement in efficiency. Indeed arbitrarily high efficiencies are achievable. We show this below.

Consider the GNE as a solution concept and assume that $\Phi \in \mathcal{L}'$ is given by linear functions as above. In particular, let c be such that $c_1 = \max_i c_i = 1$ and let the reserve price π be a number in $(0, 1)$. Let x be a GNE of this game. i.e., suppose there exists $\Lambda = (\lambda_1, \dots, \lambda_N)$ such that

$$0 \leq x \perp -c + \pi \mathbf{1} + \Lambda \geq 0$$

$$0 \leq \Lambda \perp C - \mathbf{1}^T x \geq 0$$

Since $\pi < 1 = c_1$, at equilibrium, the Lagrange multiplier λ_1 for player 1, has to be positive. The complementarity between λ_1 and ‘ $C - \mathbf{1}^T x$ ’ now ensures that $\mathbf{1}^T x = C$ holds for the GNE x . Therefore at least one component of x is positive. If x is a VE, i.e. a GNE with $\lambda_1 = \lambda_j$ for all j , those players i with $c_i = c_1$ receive positive quantity, whereas the rest receive zero. If x is a GNE, denote by \mathcal{I} the set of players with marginal utility at least π , i.e.,

$$\mathcal{I} = \{i \in \mathcal{N} \mid c_i \geq \pi\}, \quad \text{and let} \quad c_{\mathcal{I}} = (c_i)_{i \in \mathcal{I}}, \quad x_{\mathcal{I}} = (x_i)_{i \in \mathcal{I}},$$

then $x_{\mathcal{I}^c}$ which is the tuple of x_j so that $c_j < \pi$ is the tuple of $|\mathcal{I}^c|$ zeroes. Consequently, $\sum_{j \in \mathcal{I}} x_j =$

C . Therefore for this game and a GNE x ,

$$\frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)} = \frac{c^T x}{\max_{z \in \mathbb{C}} c^T z} = \frac{c_{\mathcal{I}}^T x_{\mathcal{I}}}{C} \geq \frac{\pi \sum_{j \in \mathcal{I}} x_j}{C} = \pi.$$

i.e. the efficiency of the GNE is at least π . Whereas every VE x is efficient:

$$\frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)} = \frac{c^T x}{\max_{z \in \mathbb{C}} c^T z} = \frac{c_1 \mathbf{1}^T x}{C} = 1.$$

This is true for for all $c > 0$ such that $\max_i c_i = 1$ when the reserve price $\pi < 1$. More generally, for any $c > 0$ and a reserve price π , we have

$$\inf_{x \in \text{GNE}(c)} \frac{\theta(x)}{\max_{z \in \mathbb{C}} \theta(z)} \begin{cases} \geq \frac{\pi}{\max_i c_i} & \pi < \max_i c_i, \\ = 0 & \pi \geq \max_i c_i. \end{cases}$$

When $\max_i c_i = c_1 = 1$ and $\pi \in (0, 1)$, note that π is a lower bound and the actual efficiency may in fact be greater than π . For e.g., consider a game where $\max_{j \neq 1} c_j < 1$ and $1 > \pi > \max_{j \neq 1} c_j$. For this game, all players other than player 1 are “eliminated”, i.e. they receive zero quantity at equilibrium. Therefore the efficiency of the GNE in this case is unity.

■ 5.7 Conclusions

This work considered resource allocation through a shared-constraint game from the point of view of economic efficiency. This game is relevant in the setting where an auctioneer is not available for operationalizing a mechanism. We clarified the relationship of this game with other modes of allocating resources. Shared-constraint games admit two kinds of equilibria, namely, the GNE and the VE. We considered a class of concave objective functions and found that the worst case efficiency over this class of both, the GNE and the VE, is zero. However we show that there is subclass for which the VE is always efficient but the GNE can be arbitrarily inefficient.

We then discussed remedies by which the worst case efficiency may be bounded away from zero. Specifically, we showed that utility functions with bounded gradients, alternative notions of efficiency and the imposition of a reserve price can mitigate the loss of efficiency.

Appendix A

Mathematical background

This section provides an introduction to some technical concepts that appear often in this thesis. Other concepts that are relevant only to a particular chapter are provided at appropriate places within the chapter.

■ A.1 Convex analysis

An excellent source for convex analytic theory is Rockafellar [Roc97].

A set X is a *cone* if for any point $x \in X$ and any $\alpha \in [0, \infty)$, the point αx lies in X . The point $\{0\}$ belongs to any cone. A set X is *convex* if for any $x, y \in X$ and $\alpha \in [0, 1]$ the point $\alpha x + (1 - \alpha)y$ lies in X . A function f from a convex set X to the reals is convex if its *epigraph*, i.e. the set $\{(x, y) \in X \times \mathbb{R} \mid y \geq f(x)\}$, is a convex set. Equivalently, $f : X \rightarrow \mathbb{R}$ is convex if for all $x, y \in X$ and $\alpha \in [0, 1]$, $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$. A function $f : X \rightarrow \mathbb{R}^n$ is convex if every component of f is convex. A function f is concave if $-f$ is convex. For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $f \in C^2$, convexity is equivalent to the positive semidefiniteness of the Hessian of f .

For an unbounded convex set there is a direction along which a point in the set can be translated by any magnitude without leaving the set. Indeed every point in the convex can be translated in this way along this direction. The convex set is said to recede in this direction, and the set of all such directions forms a cone, called the *recession cone*. For any closed convex set X , we let X_∞ denote its recession cone:

$$X_\infty = \{d \mid X + \tau d \subseteq X, \quad \forall \tau \geq 0\}.$$

X is bounded if and only if $X_\infty = \{0\}$.

Appendix A. Mathematical background

We now define some other cones. For any set $X \subseteq \mathbb{R}^n$ and a point $z \in X$, let $\mathcal{N}(z; X)$, $\mathcal{T}(z; X)$ denote the *normal cone* and the *tangent cone* of X at z respectively defined as follows.

$$\mathcal{N}(z; X) = \{d \mid d^T(y - z) \leq 0 \quad \forall y \in X\},$$

$$\mathcal{T}(z; X) = \left\{d \mid \exists \{\tau_k\} \subseteq (0, \infty) \text{ and } \{y_k\} \subseteq X, \text{ s.t. } \tau_k \rightarrow 0, y_k \rightarrow z, \text{ and } d = \lim_{k \rightarrow \infty} \frac{y_k - z}{\tau_k}\right\}.$$

For points z outside X , the tangent and normal cones are taken to be empty. $\mathcal{T}(z; X)$ is the cone of directions from which z can be approached from within X . $\mathcal{N}(z; X)$ is the cone of outward pointing normals to X at z . The tangent cone is always a closed set. For z in the interior of X , the tangent cone is \mathbb{R}^n and the normal cone $\{0\}$. For a convex set X and any point $z \in X$, the tangent and normal are related as follows.

$$\mathcal{T}(z, X)^* = -\mathcal{N}(z; X).$$

For any cone X , we use X^* to denote its dual cone as

$$X^* = \{d \mid d^T x \geq 0 \quad \forall x \in X\}.$$

The dual cone is always a closed and convex cone.

■ A.2 Set-valued analysis

The following information is found in Aubin [AF90] and Hogan [Hog73].

Let X, Y be metric spaces with metrics d_X and d_Y respectively and let 2^Y be the set of all subsets of Y . A set-valued map is a function $T : X \rightarrow 2^Y$. The domain of T is the set of arguments x for which $T(x)$ is nonempty:

$$\text{dom}(T) = \{x \in X \mid T(x) \neq \emptyset\}.$$

A.2 Set-valued analysis

The graph of T , $\text{graph}(T)$ and the range of T , $\text{range}(T)$, are the sets

$$\begin{aligned}\text{graph}(T) &= \{(x, y) \in X \times Y \mid y \in T(x)\}, \\ \text{range}(T) &= \bigcup_{x \in \text{dom}(T)} T(x).\end{aligned}$$

The following definition contains several notions related to the continuity of set-valued maps.

Definition A.1 *A set-valued map $T : X \rightarrow 2^Y$ is said to be*

(a) *upper semicontinuous at $x \in \text{dom}(T)$ if for any neighbourhood \mathcal{U} of $T(x)$,*

$$\exists \delta > 0 \text{ such that } \forall x' \in B_X(x, \delta), \quad F(x') \subseteq \mathcal{U}.$$

(b) *lower semicontinuous at $x \in \text{dom}(T)$ if for any $y \in T(x)$ and any sequence $\{x_k\} \subseteq \text{dom}(T)$ converging to x , there exists a sequence of elements $y_k \in T(x_k)$ converging to y .*

(c) *continuous at $x \in \text{dom}(T)$ if it is both upper and lower semicontinuous at x .*

(d) *closed at x if $\{x_k\} \rightarrow x$, $y_k \in T(x_k)$ and $y_k \rightarrow y$ implies $y \in T(x)$.*

(e) *open at x if $\{x_k\} \rightarrow x$ and $y \in T(x)$ imply the existence of a sequence $\{y_k\}$ such that $y_k \in T(x_k)$ for k sufficiently large and $y_k \rightarrow y$.*

(f) *T is upper semicontinuous (lower semicontinuous, continuous, closed, open) if it is upper semicontinuous (respectively lower semicontinuous, continuous, closed, open) at every point in its domain.*

If T is upper semicontinuous at x it is closed at x . If T is closed at x and *locally bounded* at x , i.e. if there exists a neighbourhood B of x such that the set

$$\bigcup_{t \in B \cap X} T(t),$$

is bounded, then T is upper semicontinuous at x . Instead of the term *locally bounded*, the term *uniformly compact* is often used, for obvious reasons. T is open if and only if it is lower semicon-

Appendix A. Mathematical background

tinuous. In some sources, “continuity” refers to a weaker notion which says T is continuous at $x \in \text{dom}(T)$ if it is both open and closed at x .

■ A.3 Variational inequalities

The following information is the first volume of Facchinei and Pang [FP03].

Let $V \subseteq \mathbb{R}^n$ and let $G : V \rightarrow \mathbb{R}^n$ be a function. The variational inequality $\text{VI}(V, G)$ is the following problem.

Find $x \in V$ such that $G(x)^T(y - x) \geq 0 \quad \forall y \in V.$ ($\text{VI}(V, G)$)
--

Usually, V is a closed convex set and G is a continuous function. A quasi-variational inequality (QVI) is a generalization of a variational inequality (VI). Let $L : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a set-valued map and let $G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function. The quasi-variational inequality $\text{QVI}(L, G)$ is the problem

Find $x \in L(x)$ such that $G(x)^T(y - x) \geq 0 \quad \forall y \in G(x).$ ($\text{QVI}(L, G)$)

Typically, L is closed and convex valued and G is continuous. The point x sought by $\text{VI}(V, G)$ (respectively, $\text{QVI}(L, G)$) is called a solution of the (respectively, quasi-)variational inequality and the set of all such x is denoted by $\text{SOL}(\text{VI}(V, G))$ (respectively, $\text{SOL}(\text{QVI}(L, G))$). Some special cases of variational inequalities are follows

- When $V = \mathbb{R}_+^n$, $\text{VI}(V, G)$ is equivalent to

Find $x \in \mathbb{R}^n$ such that $0 \leq x \perp G(x) \geq 0.$ ($\text{CP}(G)$)
--

$\text{CP}(G)$ is called a complementarity problem.

- When G is an affine function, $G(x) \equiv Mx + q$, the problem is called a linear complementarity problem and is denoted by $\text{LCP}(M, q)$.

Find $x \in \mathbb{R}^n$ such that $0 \leq x \perp Mx + q \geq 0.$ ($\text{LCP}(M, q)$)
--

A.3 Variational inequalities

- If

$$V = \prod_{i=1}^k V_i \quad (\text{A.1})$$

for some $V_i \subseteq \mathbb{R}^{n_i}$, $\sum n_i = n$, a VI with set V is called a cartesian VI.

The Euclidean projection of a point x on the set V is the solution of the optimization problem

$\begin{aligned} \text{P}(x) \quad & \underset{y}{\text{minimize}} \quad \ y - x\ \\ & \text{subject to} \quad y \in V, \end{aligned}$

and it has the following property.

Lemma A.1 (Property of projection) *Let $V \subseteq \mathbb{R}^m$ be a closed convex set and x be a point in \mathbb{R}^m . Then the projection of x on D , $\Pi_V(x)$, satisfies $(y - \Pi_V(x))^T(\Pi_V(x) - x) \geq 0$ for each y in V .*

The *natural map* of $\text{VI}(V, G)$, $\mathbf{F}_V^{\text{nat}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined as

$$\mathbf{F}_V^{\text{nat}}(v) \equiv v - \Pi_V(v - F(v))$$

where $\Pi_V : \mathbb{R}^n \rightarrow V$ is the Euclidean projection on V , provides an equation reformulation of the VI. Let $\tilde{\mathbf{F}}_L^{\text{nat}} : \text{dom}(L) \rightarrow \mathbb{R}^n$ denote a similar natural map for $\text{QVI}(L, G)$ defined as

$$\tilde{\mathbf{F}}_K^{\text{nat}}(v) := v - \Pi_{K(v)}(v - G(v)), \quad \forall v \in \text{dom}(L).$$

We then have:

Proposition A.2 *A vector x solves $\text{VI}(V, G)$ if and only if $\mathbf{F}_V^{\text{nat}}(x) = 0$ and x solves $\text{QVI}(L, G)$ if and only if $\tilde{\mathbf{F}}_L^{\text{nat}}(x) = 0$.*

$\mathbf{F}_V^{\text{nat}}$ is a continuous function when V is closed and convex but the continuity of $\tilde{\mathbf{F}}_K^{\text{nat}}$ relies on the continuity of the set-valued map K (see Appendix A.2).

Lemma A.3 (Lemma 2.8.2 [FP03]) *Let $x \in \text{dom}(K)$ and y be any point in \mathbb{R}^m . Then $\phi(x, y) := \mathbf{F}_{K(x)}^{\text{nat}}(y)$ is continuous at (x, y) for all $y \in \mathbb{R}^m$ if and only if $K(\cdot)$ is continuous at x .*

Appendix A. Mathematical background

Proposition 4.7.1 in [FP03, page 401] provides sufficient conditions for K to be continuous when \mathbb{C} is given by an algebraic constraint.

An equivalent formulation of the VI and QVI, in the form of a generalized equation, can given using the normal cone.

Proposition A.4 *A vector x solves $\text{VI}(V, G)$ if and only if $0 \in G(x) + \mathcal{N}(x, V)$. Likewise, x solves $\text{QVI}(L, G)$ if and only $0 \in G(x) + \mathcal{N}(x, L(x))$.*

Following are some function classes. A mapping $G : V \rightarrow \mathbb{R}^n$ is said to be

- (a) *pseudo-monotone* if for all $x, y \in V$, $G(y)^T(x - y) \geq 0 \implies G(x)^T(x - y) \geq 0$.
- (b) *monotone* if for all $x, y \in V$, $(G(x) - G(y))^T(x - y) \geq 0$. It is strictly monotone if the inequality holds strictly for all $x \neq y$.
- (c) Let V be a cartesian product of sets given by (A.1). $G = (G_1, \dots, G_k)$, $G_j \in \mathbb{R}^{n_j}$ for all j , is called \mathbf{P}_0 if for all $x = (x_1, \dots, x_k) \in V$, $x_j \in V_j$ for all j , and $y = (y_1, \dots, y_k) \in V$, $y_j \in V_j$ for all j , there exists an index $i \in \{1, \dots, k\}$ such that and

$$x_i \neq y_i \quad \text{and} \quad (G_i(x) - G_i(y))^T(x_i - y_i) \geq 0.$$

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